BASIC

GEOMETRY

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Pl'eface

This textbook in geometry differs in several essential respects from other current texts covering the same subject-matter. The nature of these differences is readily apparent to any experienced teacher who briefly scrutinizes the content of the book. In this connection, however, a word of caution is in order. Teachers of experience, from their very familiarity with other texts, may tend at first sight to be misled by the novelty of this presentation and may see in it a difficulty which is apparent rather than real. They should be quick to realize, however, that their students—unhampered by experience—will not be so bothered. Actual classroom experience with an experimental edition of these materials over a period of several years has conclusively demonstrated their teachability. From these materials students acquire a mastery of geometry that is noticeably superior to the mastery gained from the traditional handling of the subject. Moreover, students trained according to the principles of this book need have no fear of college entrance examinations in geometry.

The traditional approach to demonstrative geometry involves careful study of certain theorems which the beginner is eager to accept without proof and which he might properly be led to take for granted as assumptions or postulates. Such an approach obscures at the very outset the meaning of "proof" and "demonstration." The employment of superposition in the proof of some of these theorems is even more demoralizing. This method of proof is so out of harmony with the larger aim of geometry instruction that despite its validity its use is commonly restricted to those few cases for which no better method can be found. For fundamental postulates of our geometry, therefore, we have chosen certain propositions of such broad import that the method of superposition will not be needed. We utilize only five fundamental postulates. They are stated and discussed in Chapter 2.

For a rigorous mathematical presentation of the postulates which we have employed, see the article "A Set of Postulates for Plane Geometry, Based on Scale and Protractor" published
in the Annals of Mathematics, Vol. XXXIII, April, 1932. Naturally it has been advisable in an elementary course to
slur over or ignore some of the subtler mathematical details, for these are not suitable material for the mind of the student
at this juncture. Nevertheless, wherever the presentation involves a substantial incompleteness, its nature is indicated so
far as possible in an accompanying footnote.

Another bugbear to beginners in geometry is "the incommensurable case." Euclid could not ignore the diagonal of the
unit square and other similar lines, though he had no numbers with which to designate their lengths. By means of inequalities
and an exceedingly shrewd definition of proportion he was able, however, to handle these incommensurable cases. Within
the last century this treatment of incommensurables—or an equivalent statement—has been taken as the definition of ir-
rationals; numbers, including the irrationals, and so avoids explicit mention of the incommensurable case. For the teacher’s convenience we present a brief discussion of the fundamental properties of the system of real numbers in the section Laws of Number, pages 284-288.

Taking for granted these fundamental properties of number also leads to many other simplifications and gives us a
tool of the greatest power and significance. Among other things it enables us to combine the ideas "equal triangles" and
"similar triangles" in general statements in which equality is but a special case of similarity, where the factor of propor-
tionality is 1. And, further, we can base the treatment of parallels on similarity. This procedure, although the reverse
of Euclid’s, is logically equivalent to it. Thus our geometry, though differing in many important respects from Euclid’s, is
still Euclidean; the differences reflect the progress of mathematics since Euclid’s time.

These changes are all by way of simplification and condensation. As a result, we have a two-dimensional geometry
built on only five fundamental postulates, seven basic theorems, and nineteen other theorems, together with seven on loci.
The increase of knowledge and the growing demands of civilization make it more important than ever before that our instruction be as compact and profitable for the student as possible.

Incorporation of the system of real numbers in three of the five fundamental assumptions of this geometry gives these as-
swnptions great breadth and power. They lead us at once to the heart of geometry. That is why a geometry that is built on so strong a base is so simple and compact. It is because of the underlying power, simplicity, and compactness of this geometry that we call it basic geometry.

In a course in demonstrative geometry our prime concern is to make the student articulate about the sort of thing that hitherto he has been doing quite unconsciously. We wish to make him critical of his own, and others', reasoning. Then we would have him turn this training to account in situations quite apart from geometry. We want him to see the need for assumptions, definitions, and undefined terms behind every body of logic; to distinguish between good and bad arguments; to see and state relations correctly and draw proper conclusions from them. To this end we have inserted analyses or summaries of the reasoning employed in the proofs of nearly all the propositions and have included in Chapters 1 and 10 a detailed consideration of logical reasoning in fields outside of geometry.

This book is designed to require one year of study, although it may readily be spread over two years in schools following that plan. Essentially it is a course in plane geometry. Realizing, however, that most students will give not more than one year to the study of geometry, we have incorporated certain material from three-dimensional geometry and from modern geometry (so called), so that for all classes of students this first year of geometry may be as rich and enduring an experience as possible. The three-dimensional material is based on the student's intuition and is not intended to be logically complete.

The exercises are important for their content and for the development of the subject. None should be omitted without careful consideration. Those marked with a star are of especial importance.

The basic theorems 6, 7, 8, 11, and 13 can be deduced from the fundamental assumptions; but they will probably appear to the student to be sufficiently obvious without proof. It will be wise, therefore, to permit the student to assume these theorems at the outset. He should return to them later, when he has caught the spirit of the subject more fully and can be interested in the problem of reducing his list of assumptions to a minimum. This comes about naturally in Chapter 10, where the assumptions are reconsidered.
The brief treatment of networks and the slope and equation of a line in Chapter 4, together with the related exercises, and the brief treatment of the equation of a circle in Chapter 5 are included to show the relation of this geometry to analytic geometry. They may be omitted without injury to the logical development of the subject. The teacher may wish to deal lightly with continuous variation in Chapter 8 and to omit in Chapter 9 the sections devoted to power of a point, radical axis, inversion, and projection.

In every class in geometry it is excellent procedure to elicit suggestions from the students as to theorems worth proving and the best way of proving them. Unfortunately, however, theorems cannot always be proved in the order proposed by the students; or they can be proved in the suggested order only if certain other intermediary theorems are interpolated at the right places. Correct decisions on questions of this sort sometimes require complete grasp of the entire logical framework of the geometry. In order to avoid confusion, therefore, the authors have chosen the order of the theorems in the several exercises; they wish, nevertheless, to approve every well-considered effort on the part of teachers to elicit the cooperation of students in building the geometry. It is not impossible that a faulty procedure suggested by the students will have more educative value than a correct procedure imposed by author or teacher. This is most likely to be true when the teacher points out to the students wherein their procedure is wrong.

The value of demonstrative geometry as prevailing taught in secondary schools is being questioned, and not without cause. It would be difficult to prove that the study of the subject necessarily leads in any large measure to those habits, attitudes, and appreciations which its proponents so eagerly claim for it. But it would be even more difficult to prove that other subjects of instruction can yield these outcomes as easily and as surely as can demonstrative geometry in the hands of an able and purposeful teacher. Teachers of demonstrative geometry are confronted with the challenge to re-shape their instruction so that it more nearly achieves the desired objectives. This textbook is offered as an instrument to that end.

GEORGE DAVID BIRKHOFF
RALPH BEATLEY
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CHAPTER 1

Reasoning. The Nature of Proof

Consider the following two disputes, just as if you had been asked to decide them.

Dispute 1. The upper picture on the opposite page shows the Mohawks and the Wildcats playing baseball. At the end of the fourth inning the score stands: Mohawks, 6; Wildcats, 7. Luke Tobin is pitching for the Mohawks. His older brother John considers himself to be about as good a pitcher as Luke, but the rest of the Mohawks think that Luke is a much better pitcher than John. They admit, however, that except for Luke they have no one who can pitch as well as John. John hits well but is not good at fielding batted balls. The rest of the team think he belongs out in left field, and that is where he is playing in this game against the Wildcats.

The two teams have only two bats, one belonging to Luke and the other to John. In the first half of the fifth inning, with two men out, the Mohawks get another run, tying the score. Then John hits a home run, putting the Mohawks ahead. The next batter hits a foul and in so doing breaks Luke’s bat. Immediately John seizes his own bat and announces to his teammates that if they don’t let him pitch the second half of the fifth inning and all the rest of the game, he will take his bat and go home.
The Mohawks greet this announcement with boos and jeers, but John sticks to his point and is perfectly willing to break up the game if he cannot have his way. He is big enough to get away with it, too, especially since he has possession of the bat.

Hank Cummings, shortstop and captain of the Mohawks, suggests that Luke and John swap positions, but that John agree to return to left field and let Luke pitch in case the Wildcats tie the score. The rest of the Mohawks think that their lead of one run is none too secure with Luke pitching, and that to let John pitch is practically to give the game to the Wildcats, even if John agrees to Hank's proposal. They declare that they would rather quit playing then and there than continue under such an agreement.

The Wildcats join the hubbub at this point and offer to punch a few faces if they don't get their turn at bat in the second half of the fifth. They announce that if the game breaks up, they will have won with a score of 7-6; that the Mohawk runs in the first half of the fifth don't count unless the Wildcats have their chance at bat in the second half of the inning.

Remember that John has just scored the run putting the Mohawks ahead and that he feels confident he can hit anything the Wildcat pitcher can offer. Remember also that the Mohawks usually depend upon Luke and John to supply not only the bats but also the ball whenever they play. Remember finally that they never let John pitch in any of their games, although they regard him as relief pitcher for Luke. Where are the right and the wrong in this argument, and “how much ought who to yield to whom?”

Dispute 2. Jane and her younger sister Mary take turns washing the dishes, one girl washing one night...
and the other the next. Last night Mary was sick; so Jane washed in her place. The girls agreed that tonight Mary would take Jane's turn. In the middle of supper, however, Donald Ames telephoned and invited Mary to go to the movies. Mary, intending to make it up to Jane by washing the dishes the next three nights in a row, accepted the invitation without consulting Jane. Jane was furious. Probably she was a bit jealous that her younger sister had a date and she had none. But to have Mary disregard her agreement without even asking Jane to release her was just too much. "How did you know I wasn't going out myself?" asked Jane. "As soon as supper was over, I was going down to Laura's. I think you ought to call Donald up this minute and tell him you can't go. Don't you, Mother?"

Now consider the following facts. Jane made no fuss about taking Mary's turn when she was sick. Her plan to visit Laura was not made and not announced until after Donald telephoned. Donald is a fine boy, just the sort Mary's mother approves. Neither girl has been led to believe that she can go out at night without her parents' permission. Where are the right and the wrong in this argument? Which girl ought to wash the dishes?

It will not be surprising if you are puzzled as to the decision you ought to give in each of these disputes; for in arguments concerning intricate personal relations it does not usually happen that one side is wholly right and the other wholly wrong. We must learn, however, to think straight in just such tangled situations as these. If you consider these situations too childish, just analyze a few industrial disputes or political disputes or disputes between nations and see if they do not all have much in common with the disputes that arise every day among boys and girls.
In order to meet the demands of our daily lives, we must know how to argue and how to prove things. In order not to mislead others or to be misled ourselves, we must be able to distinguish between good and bad reasoning. And, finally, we not only must learn to appreciate the logical connections of the several parts of a single restricted topic or argument, but also must have some practice and training in analyzing the logical structure of a protracted argument or of a series of related arguments.

It is difficult, however, to learn all these things from situations such as occur in everyday life. What we need is a series of abstract and quite impersonal situations to argue about in which one side is surely right and the other surely wrong. The best source of such situations for our purposes is geometry. Consequently we shall study geometric situations in order to get practice in straight thinking and logical argument, and in order to see how it is possible to arrange all the ideas associated with a given subject in a coherent, logical system that is free from contradictions. That is, we shall regard the proof of each proposition of geometry as an example of correct method in argumentation, and shall come to regard geometry as our ideal of an abstract logical system. Later, when we have acquired some skill in abstract reasoning, we shall try to see how much of this skill we can apply to problems from real life.

**DEFINITIONS AND UNDEFINED TERMS**

Sometimes we find it hard to explain to someone else a subject that is very clear to us. The trouble often is that some of the words we use have a meaning for the other person different from the meaning they have for us. To avoid misunderstanding, it is important that we carefully define every word that we use, so far as this is possible.
Since a good definition uses only ideas that are simpler than the idea that is being defined, some of the simplest ideas must remain undefined. Thus we may define a "sickle" as a "curved knife used for cutting grass," and we may define a "knife" as a "sharp-edged tool used for cutting." If there is someone who does not know what a "tool" is, or what "to cut" means, he will never know what a knife or a sickle is. It will not help to try to define a "tool" as an "implement," or "to cut" as "to sever, to inflict an incision on." For we should then have to explain the meaning of "implement," "sever," "inflict," "incision"; and in defining these we should need the words "tool" and "cut" again. Words of this sort, therefore, must remain undefined, along with such words as "the," "but," "and," "although."

We must be careful not to confuse everyday colloquial meanings of words with precise meanings. For example, to most people the expression "he went straight across the street" means that he crossed the street at a right angle to the curb; but the expression might also mean that he crossed the street at any angle to the curb provided he crossed in a straight line. We shall be obliged, therefore, to be very careful in our use of language and to say exactly what we mean; and in cases where there is any doubt about the meaning of a word we should agree in advance what that word shall mean for us.

In talking about ideas that usually we take for granted, we may become confused because we cannot find words to describe them precisely. In such instances we may find certain technical terms helpful. As an illustration of this let us examine Fig. 1 on the next page. It contains a triangle $AB\,C$ and a second triangle $A'B'C'$. Fig. 1 shows triangle $A'B'C'$ in five different positions. In the two triangles side $AB$ equals side $A'B'$; side $AC$ equals side $A'C'$; and angle $A$ equals angle $A'$. Let us agree that
whenever two triangles contain equal sides and angles in just this way, no matter how one triangle may be placed with respect to the other, then the two triangles are equal. Now try to describe these relationships accurately in general terms, that is, without using the letters \( A, B, \) and \( C \). Probably your description will be a bewildering mass of words unless you hit upon some short way of describing the equal parts of the triangles in their proper relation to each other. By using the words "included" and "respectively" you can describe the relationships briefly and accurately as follows: If in two triangles two sides and the included angle of one are equal respectively to two sides and the included angle of the other, the two triangles are equal. Though technical terms like "included" and "respectively" are sometimes very helpful, you need not feel obliged to use such technical terms in your own work. Any wording that is accurate is acceptable. All that is necessary is that the words and phrases used shall have the same meaning for everybody.

Where there is a choice of two expressions, each equally simple, we should always choose the more accurate one. Often we must choose between a simple expression and a more involved expression that is also more accurate. In such cases we may use the simple expression of everyday speech so long as there is no danger of confusion. For
example, the word "circle" means strictly a curved line (in a plane) every point of which is equidistant from a fixed point called the center of the circle. The word "circle" does not mean the distance around the circle nor that portion of the plane enclosed by the circle. It is strictly accurate, therefore, to say that a certain line—for example, a diameter—cuts a circle in but two points, for neither the center of the circle nor any other point inside the curved line is a point of the circle. The word "circumference" means the distance around the circle, as distinguished from the circle itself; it is inaccurate, therefore, and quite unnecessary to speak of a line as cutting the circumference. It is equally inaccurate to speak of the area of a circle, for the curved line has no area at all. In this case, however, there is no possibility of confusion, and so we shall use the expression "area of a circle" because it is so much shorter than saying "the area of the surface enclosed by the circle."

The word "diameter," strictly speaking, means a straight line through the center from one point on a circle to another. We also use the word to mean the length of such a line. But we shall do this only when there is no likelihood of confusion. Similarly, the word "radius" may mean either a straight line from a point on a circle to its center or the length of such a line.

Now let us think about the following facts and see how they may be described in general terms. John is the same height as Henry, and Bill is the same height as Henry; therefore, John and Bill are the same height. The technical expression that describes any such relationship is: "Things equal to the same thing are equal to each other."

The exercises on pages 16 and 17 will test your ability to state ideas accurately. You may need to use a dictionary to find the exact meaning of some of the words.
EXERCISES

1. Does the phrase "straight up" mean the same thing in the two following sentences?
   (a) He went straight up the hill.
   (b) He looked straight up over his head.

   While these statements are sufficiently accurate for everyday use, they might easily be open to question if used in an argument in court or in a geometric proposition. Can sentence (b) have more than one meaning? Rewrite sentences (a) and (b) so that they can have but one definite meaning.

2. Which is heavier, a quart of "light cream" or a quart of "heavy cream"? Suggest a better pair of adjectives that might be used to distinguish "light cream" from "heavy cream."

3. Does sugar "melt" in hot coffee, or does it "dissolve"? What happens to a lump of butter when it is put into hot oyster stew? What substances can really "melt in your mouth"? Gasoline is often used to clean the thin film of oil from the inside of an empty oil can. Does the oil melt or dissolve in the gasoline?

4. Distinguish between "He is mad" and "He is angry."

5. What is an "ambiguous" statement?

6. In what way is the following statement ambiguous? "Nothing is too good for him!"

7. Certain politicians seeking re-election requested the backing of their chief in the form of a public statement from him. He complied as follows: "They have never failed me." Is this an unqualified recommendation, or is it open to another interpretation?

8. Is south the opposite of north?
9. Two men are at the North Pole. One of them goes south; the other goes in the opposite direction. In what direction does the latter go?

10. Is a plumb line vertical or perpendicular?

11. Can a single line ever be perpendicular?

12. If two lines are perpendicular, must one be vertical and the other horizontal?

13. Can three lines meet in a single point in such a way that each line is perpendicular to each of the others?

14. Is a "flat surface" the same thing as a "plane surface"?

15. Do the expressions "Lay it down flat" and "Lay it down so it is horizontal" mean the same thing?

16. Must a flat surface be also level?

ASSUMPTIONS

We have seen how easily we may become confused in our thinking if we attempt to discuss an idea without clearly defining it. We have seen also that every idea we define depends upon simpler ideas, and that a few of the very simplest ideas must remain undefined. These ideas that we must either define or else leave undefined are represented for the most part by single words. Sometimes they are represented by phrases, but they are never complete sentences. We can combine these undefined and defined words into sentences, or statements, that are either true or false. Such statements are often called propositions. For example, each of the statements "4 times 8 is 32" and "4 times 8 is 28" is a proposition.

Often you will hear two people discussing a subject—that is, a proposition—each arguing logically but without persuading the other to alter his stand in the least. Usu-
ally neither has attempted to discover the earlier propositions that are being taken for granted in his opponent’s argument, and neither is aware of the earlier propositions on which his own argument depends. Even the most familiar propositions of arithmetic are based on earlier propositions that we are in the habit of overlooking. We ordinarily consider that \(4 \times 8 = 32\) is a true statement; but it is true only so long as we take for granted that the base of the number system we are using is 10. If we were to use the number system whose base is 12, we should find that the product of 4 and 8 would be written 28, because 28 in such a system means two 12’s plus 8. In the number system whose base is 12 the statement \(4 \times 8 = 32\) is false.

The first step in discussing a proposition with a view to proving it true or false is to discover the earlier propositions upon which it depends. A few of these earlier propositions must remain unproved and be taken for granted, for we must have some place from which to begin our chain of reasoning. We cannot prove everything. The propositions that we take for granted without proof are called assumptions. When we criticize an argument or assert that a certain proposition is true or false, we should first discover the assumptions upon which the argument or proposition depends.

A given set of assumptions may seem very strange; but so long as no one of them is in contradiction with the others, every proposition that follows logically from them will be true with respect to them. When we say that a certain proposition is true, we mean merely that the proposition follows logically from the assumptions upon which it is based. If we grant the assumptions, we must admit the truth of the proposition. With regard to a different set of assumptions the same proposition may, or may not, be true. The assumptions themselves are neither true...
nor false. They may be said to be "true" only in the sense that their truth has been assumed.

Neither in mathematics nor in life around us can every proposition be proved. Some few must remain unproved and be taken for granted. We may assume any propositions we like provided they are all consistent; and we may prove from these propositions any others that we can. Propositions that can be logically deduced from the assumptions are often called theorems. The propositions that are chosen as assumptions are sometimes called postulates, or axioms. The three words "assumption," "postulate," "axiom" all have the same meaning. Assumptions are related to theorems in the same way that undefined terms are related to definitions: we cannot define every idea, nor can we prove every proposition.

EXERCISES

1. Assume that lead is heavier than iron, and that lead melts at a lower temperature than iron. Which of the following propositions must then be true?
   (a) Solid iron floats on molten lead.
   (b) Solid iron sinks in molten lead.
   (c) Solid lead floats on molten iron.
   (d) Solid lead sinks in molten iron.

2. Assume that iron is heavier than lead, and that iron melts at a lower temperature than lead. Which of the foregoing propositions must then be true?

3. Assume that lead is heavier than iron, and that iron melts at a lower temperature than lead. Which of the foregoing propositions must then be true?

4. Assume that iron is heavier than lead, and that lead melts at a lower temperature than iron. Which of the foregoing propositions must then be true?
5. Assume that ludge is heavier than runk, and that runk melts at a lower temperature than ludge. State a proposition that follows logically from these assumptions, or, in other words, that is true with respect to them.

6. Should we call $4 + 4 = 13$ an utterly false statement, or should we say that it would be true if the base of the number system were 5?

7. Salesman: "You should buy our weather-strip because recently, in competition with eleven others, it passed a rigorous government test, with a rating of 93% efficient." What assumptions does this salesman wish the buyer to make?

THE NATURE OF GEOMETRIC PROOF

Let us now try to prove some geometric propositions using only the following three assumptions.

ASSUMPTION 1. If in two triangles two sides and the included angle of one are equal respectively to two sides and the included angle of the other, the two triangles are equal.

ASSUMPTION 2. If in two triangles a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other, the two triangles are equal.

ASSUMPTION 3. Through a given point there is one and only one perpendicular to a given line.

Before you try to use these assumptions in proving any propositions, be sure that you agree with other students as to the meaning of all the terms used in these assumptions. The term "triangle" is defined on page 56, and the term "perpendicular," on page 50. You must also understand the terms "bisect" and "mid-point," as defined on page 44.
Now let us try to prove the following theorem. We shall call it Theorem A.

**Theorem A.** If two sides of a triangle are equal, the angles opposite these sides are equal.

**GIVEN:** Triangle $ABC$ (Fig. 2) in which $AB = AC$.

**TO PROVE:** $\angle B = \angle C$.

**ANALYSIS:** Since Assumptions 1 and 2 afford the only ways of proving lengths or angles equal, we must arrange to get two triangles into the diagram involving angles $B$ and $C$ separately. Drawing $AD$ perpendicular to $BC$ will do this, but our assumptions do not enable us to prove triangles $ABD$ and $ACD$ equal. If, instead, we draw $AD$ so that it bisects angle $BAC$, we can complete the proof.

**PROOF:** Draw $AD$ so that it bisects angle $BAC$. Then in triangle $ABD$ and triangle $ACD$,

- $AB = AC$ (Given),
- $\angle BAD = \angle CAD$ (By construction),
- $AD = AD$ (Given),

and so triangle $ABD = triangle ACD$ (by Assumption 1), and the corresponding parts of these triangles are equal. In particular, $\angle B = \angle C$, which is what we set out to prove.

By adding to our list of definitions and assumptions we could also show that $AD$ is perpendicular to $BC$.

It is not necessary to set down every proof in this form; this will serve, however, as a helpful model. In proving theorems it is important to keep clearly in mind exactly what is given and what is to be proved, and to distin-

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*The symbol $\angle$ means angle.*
guish between these two; so any form which encourages this is desirable. Note that what is given and what is to be proved are stated in terms of the diagram.

That which is given is sometimes called the HYPOTHESIS, and that which is to be proved is sometimes called the CONCLUSION. In the statement of the theorem the hypothesis is often found in a clause beginning with "if" or "when" or the like, or the first part of the hypothesis is found in such a clause; the conclusion is then usually the remainder of the statement. Point out the hypothesis and the conclusion in Theorem A.

Sometimes Theorem A is stated in the form of a simple declarative sentence: The angles opposite the equal sides of an isosceles triangle are equal. In this form it is much harder to determine the hypothesis and the conclusion. The hypothesis is now hidden in a part of the subject of the sentence, in the phrase "of an isosceles triangle." These four words indicate that a triangle with two equal sides is given. The conclusion, shown in italics, is divided between the subject and the predicate. If you have difficulty in determining the hypothesis and the conclusion of theorems worded in this way, you will find it helpful to restate them in the "If . . . , then . . . " form.

Theorem B. If two sides of a triangle are equal, the line which bisects the angle between the equal sides bisects the third side.

GIVEN: Triangle ABC (Fig. 3) in which \( AB = AC \) and \( \angle BAD = \angle CAD \).

TO PROVE: \( BD = DC \).

ANALYSIS: Here we already have two triangles involving \( BD \) and \( DC \).
Prove that these triangles are equal. You may use any of the three assumptions and also Theorem A.

In Theorems A and B we have assumed, without saying so, that the line which bisects angle $A$ actually meets line $BC$ between $B$ and $C$. This seems quite obvious as we look at Figs. 2 and 3 and hardly worth mentioning. To be strictly logical, however, we ought not to assume this without listing it with our other assumptions. It has no special standing because it appears to be obvious. For the sake of simplicity, however, we usually ignore such logical refinements. We mention this one here in order to emphasize the distinction between statements that our experience says are "obvious" and statements that are logically "true." For occasionally we may work with assumptions like those in Exercises 2-4 on page 19, which are directly contrary to our experience. Strictly, we may recognize as "true" only the propositions that we expressly assume and the theorems that can be derived from them.

**Theorem C.** If two sides of a triangle are equal, the line joining the corner (or vertex) between the equal sides and the mid-point of the third side bisects the angle between the equal sides.

**Given:** Triangle $ABC$ (Fig. 4) in which $AB = AC$ and $BM = MC$.

**To Prove:** $\angle BAM = \angle CAM$.

**Analysis:** Here we have two triangles involving angles $BAM$ and $CAM$, in which three sides of one triangle are equal respectively to three sides of the other triangle. We have no assumption and no theorem as yet to tell us that under these conditions two triangles are equal. But one of the theorems we have just proved tells us enough to prove these triangles equal.

Complete the proof.
EXERCISE 1

1. In Fig. 5 line PM is perpendicular to line AB at its mid-point M. Prove that PA = PB.

2. State a theorem suggested by Fig. 6, in which \( AB = AC \) and \( BD = CD \). Your statement of the theorem may be in terms of the diagram. Prove your theorem.

3. State a second theorem suggested by Fig. 6 and prove it.

4. State a theorem suggested by Fig. 7, in which \( AB = BD \) and \( AC = CD \). Prove your theorem.

5. Point out what is given and what is to be proved in each of the following propositions. You can do this without knowing the meaning of every term.

(a) If two angles of a triangle are equal, the sides opposite these angles are equal.
If a quadrilateral has three right angles, its fourth angle is a right angle also.

If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

A radius perpendicular to a chord of a circle bisects the chord.

The sum of the angles of a triangle is 180 degrees.

If Bobby was alone in the house between 3 and 4 in the afternoon, then it was he who took the cookies out of the cake box.

If your dog is barking, there is a stranger on the premises.

6. Reword the following propositions in the "If ..., then ..." form.

(a) Equal chords of a circle are equally distant from the center of the circle.

(b) The opposite angles of a parallelogram are equal.

(c) A hungry baby cries.

What have we accomplished so far? We have acquired a notion of what it means to prove a proposition and have recognized the need for certain assumptions, definitions, and undefined terms. But the three assumptions on page 20 are not enough to enable us to prove the ordinary theorems of elementary geometry. So we shall begin again in Chapter 2 with five new assumptions which we shall adopt as the basis of our geometry. We shall indicate the terms we shall take as undefined and shall introduce definitions of other terms as we require them. In the later chapters we shall consider and prove a large number of important and interesting geometric propositions, remembering that each new one we prove may be used to help in the proof of some later proposition. There will be opportunity also to apply our increasing ability in demonstration to propositions from real life.
Before we turn to this fuller treatment of geometry, however, it will be helpful to consider one or two interesting questions of logic which are just as important for the arguments and discussions of everyday life as for propositions in geometry.

**CONVERSE PROPOSITIONS**

Consider the two propositions that are given below. If you interchange the hypothesis and the conclusion in the first proposition, you get the second one. Notice that you do this by interchanging the ideas rather than the precise words of the hypothesis and the conclusion.

1. If two sides of a triangle are equal, the angles opposite these sides are equal.
2. If two angles of a triangle are equal, the sides opposite these angles are equal.

The second proposition is called the CONVERSE of the first, and the first proposition is likewise the converse of the second. Each proposition is the converse of the other.

Frequently, if a proposition has been proved true, its converse will also be true. You cannot assume, however, that this is always the case. Sometimes the converse will prove to be false. Assuming the truth of a converse is one of the chief causes of faulty reasoning, as the following incident shows.

On the evening that a boat sailed from Liverpool an English lady came aboard and went directly to her cabin without meeting any of her fellow passengers. The next morning she complained to the deck steward that she could not get to sleep till long after midnight because of "those noisy Americans" right outside her cabin. She jumped to the conclusion—not necessarily true—that since some Americans call attention to themselves by
their noisy conduct while traveling, all who do so must be Americans.

The converses of the following propositions are not true.
1. If a straight line goes through the center of a circle, it cuts the circle in two distinct points.
2. If a circle is cut by two parallel lines, the arcs between the lines are equal.
3. If you do not leave home on time, you are late to school.
4. Every point on one rail of a street-car track is 17 feet from the trolley wire.

We may state the converse of the fourth proposition as follows: Every point that is 17 feet from the trolley wire lies on one rail of the car track. This is not true because there are many different points on an overhanging elm tree which are also 17 feet from the trolley wire.

Very often a proposition is so worded that it requires thought to state the converse proposition correctly. This is true especially when a word or phrase belongs properly both to the hypothesis and the conclusion but actually appears in only one of these, or in neither. In all such cases it is helpful to state the original proposition in the form "If A . . ., then B . . .," adding or repeating phrases as needed, from which it is easy to write the converse proposition "If B . . ., then A . . . ."

Consider, for example, the proposition "If two sides of a triangle are equal, the angles opposite these sides are equal." Here the triangle idea obviously belongs to the conclusion as well as to the hypothesis. Again, in the proposition "The diagonals of a parallelogram bisect each other," the quadrilateral idea belongs to both hypothesis and conclusion although mentioned in neither. The latter proposition and its converse can be stated as fol-
If a quadrilateral is a parallelogram, the diagonals of the quadrilateral bisect each other; and conversely, if the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

EXERCISES

Write the converses of the following propositions and state in each case whether you think the converse is true or false.

1. If all three sides of a triangle are equal, the three angles of the triangle are equal also.
2. If all four sides of a quadrilateral are equal, the quadrilateral is a parallelogram.
3. If two triangles are equal, the angles of the two triangles are respectively equal.
4. If two rectangles are equal, the diagonals of one rectangle are equal to the diagonals of the other.
5. If dew has fallen, the grass is wet.
6. If the milkman has come, there are three bottles of milk on the back porch.
7. If Aunt Marian is coming, we shall have waffles for supper.
8. If the tree is dead, it has no sap in it.
9. A squirrel is an animal having a thick bushy tail.
10. A hungry baby cries.
11. Every point in line \( AB \) is a point in line \( ABC \).

A proposition and its converse are often combined in one statement by means of the expression "If and only if," or equivalent phrasing. When you meet such a proposition, first separate it into its two component
propositions. For example, you may separate the statement "Two angles of a triangle are equal if and only if two sides of the triangle are equal" into these two propositions:

1. If two sides are equal, two angles are equal.
2. If two angles are equal, two sides are equal.

The proposition "Two angles of a triangle are equal only if two sides are equal" is another way of saying "If two angles are equal, two sides are equal." In other words, the propositions "Only if A . . ., then B . . ." and "If B . . ., then A . . ." are equivalent. That is why the proposition "If A . . ., then B . . ." and its converse "If B . . ., then A . . ." can be compressed into the single statement "If and only-if A . . ., then B . . .". These ideas are discussed again in Chapters 3 and 9. You need not puzzle over them any more now. You will learn that the proof of a statement beginning "If and only if . . ." requires the proof of two theorems, one of which is the converse of the other.

**FAULTY REASONING**

In proving geometric propositions you must always be alert to avoid faulty reasoning. The technical term for faulty reasoning is the Latin phrase *non sequitur*, meaning *It does not follow*. "If it is 10 feet from A to B and 7 feet from B to C; therefore it is 17 feet from A to C" is a *non sequitur*: It does not follow. There is nothing in the statement to show that A, B, and C are in a straight line; if they are not, then the distance from A to C will be less than 17 feet. Again, to argue that either John or Henry broke my front window, because I saw them playing ball in the street in front of my house at 3 P. M. and found my window broken at 5 P. M., is also a *non sequitur*. They may have done the damage, but I cannot
prove it without more evidence. As already pointed out, one of the chief sources of faulty reasoning is the confusion of proposition and converse.

**EXERCISES**

Point out the errors in reasoning in these exercises:

1. All roads going west lead to Indiana. This road goes south and therefore does not lead to Indiana.

2. Margaret: "I think your answer to the fifth problem is wrong."
   Helen: "Well, I guess I ought to know. Didn't my mother spend an hour on it last night?"

3. Mother: "No, David, you have had all the ice-cream that is good for you."
   David: "But, Mother, when I had typhoid fever, the doctor said I could have all the ice-cream I wanted."

4. "There, now, the cabbage has burned and stuck to the kettle! I never did have any use for a gas stove!"

5. "For the past nine years I have had dizzy spells and pains in my back. Recently I bought a bottle of your Liquid Panacea and have not had a sick day since."
   **ANSWER:** The fact that he was taking something for his trouble may in itself have helped the man to forget his aches and pains. Or possibly this particular patent medicine may actually have deadened his sensibility to pain for the time being, while all the time laying the foundation for more serious trouble later on.

6. The drinking of tea is a strain on the system and shows itself in deep lines on each side of the mouth. For it is well known that Arctic explorers drink a great deal of tea when in the frozen north, and they all have those deep lines around their mouths.
7. A man proved to his own satisfaction that changes in temperature have no effect on the length of a steel bar. For, measuring the bar with a steel ruler on the hottest and coldest days of the year, he found the length of the bar to be always the same.

8. Army tests during the World War of 1914-1918 showed that seventy-nine per cent of the people of the United States are below average mentally.

9. "Let's see. We won from Vernon 15 to 0, and from Aggie 9 to 2; we beat the Navy 5 to 4, and lost to Yale 2 to 3. Just as I thought—a bit over-confident!"

10. A traveler reported that a coin had recently been unearthed at Pompeii bearing the date 70 B.C.

Another common error in reasoning is called "begging the question" (in Latin, petitio principii). A glaring example of this occurs in the following attempt to prove the proposition that if one side of a triangle is bisected by the perpendicular from the opposite vertex, the other two sides of the triangle are equal.

GIVEN: Triangle \( \triangle ABC \) (Fig. 8) in which \( CD \) is perpendicular to \( AB \), and \( AD = DB \).

TO PROVE: \( AC = BC \).

PROOF: In triangles \( \triangle ACD \) and \( \triangle BCD \),

\[ \angle ADC = \angle BDC, \]

because \( CD \) is perpendicular to \( AB \),

\[ AD = DB \] (Given),

\[ \angle CAD = \angle CBD, \]

because if two sides of a triangle are equal, the angles opposite these sides are equal.

Therefore triangle \( \triangle ACD \) equals triangle \( \triangle BCD \), because two triangles are equal if a side and two adjacent angles
of one are equal respectively to a side and two adjacent angles of the other.

Therefore \( AC = BC \), being corresponding parts of equal triangles.

The fault in this argument is in the third step. In this step we assume that we have a triangle with two sides equal. This, however, is what we are trying to prove. Such an error is called "begging the question."

There is a logical proof for this theorem. Complete the proof correctly.

We know how illogical it is to beg the question in an argument, and we avoid it in all obvious instances. Usually when we do this sort of thing the error is not so easily detected as in the example above. We shall have to be very careful, therefore, to make sure that in our proofs we are not making use-even though quite indirectly-of the proposition we are trying to prove.

Sometimes it is the diagram and not the argument that is at fault. Either the diagram does not fulfill all the conditions stated in the proposition, or a construction line has been drawn in a misleading way. In that case the reasoning in the proof may be correct yet prove nothing so far as the proposition is concerned, because the reasoning is based on an incorrect diagram. Plausible but incorrect constructions underlie some of the well-known geometric fallacies. For example, the diagram in Fig. 9 is often used to "prove" the proposition that all triangles are isosceles. In this diagram the line perpendicular to \( AB \) at its mid-point \( M \) must meet the bisector of angle \( ACB \) at some point \( O \). The dia-
gram is distorted slightly to make 0 fall inside the triangle; and if this is accepted, the rest of the argument—what is, not given here—is valid. Actually, however, this point 0 belongs outside the triangle, and the fallacy is revealed.

Even an accurate construction may sometimes mislead us. Who would guess that in Fig. 10 the lines $AB$ and $CD$ are parallel or that $EF$ and $GH$ are equal? It is not safe to trust appearances.

$$A = \overline{AB} \quad C = \overline{CD}$$

Fig. 10

In drawing diagrams to help prove theorems we must be careful to make the figures as general as possible. If we are asked to prove something about a triangle, then we should not draw a triangle with two sides apparently equal. The phrase “a triangle” means “any triangle.” If two sides of the triangle in our diagram appear to be equal, we may be led by this diagram to prove the proposition only for the special case of an isosceles triangle instead of for the general case of any triangle.

**INDIRECT METHOD**

Logically it is just as convincing to prove that a conclusion cannot be wrong as to prove directly that it must be right. When, therefore, we find ourselves unable to prove directly that a certain conclusion is right, we can sometimes do what is equivalent, that is, show that all other possible conclusions are wrong. The following incident illustrates this method of reasoning.

I am awakened during the night by the clock while it
is striking. I hear the clock strike only six times, but by the following reasoning I know that it is just eleven o'clock. It cannot be 9 or 10 o'clock, because there is a light in the Swains' house. Their house has been closed for a week while they were out of town, and they were coming back on the 10:15 train tonight. It cannot be midnight, because I just heard a street-car go down the hill, and the last car goes by here at 11:30.

Now let us see how such reasoning may be used in a geometric proof. Suppose, for example, that we have already proved that if two sides of a triangle are unequal, the angles opposite these sides will be unequal and the greater angle will be opposite the greater side; and suppose further that we are now asked to prove the converse of the above, namely, that if two angles of a triangle are unequal, the sides opposite these angles will be unequal and the greater side will be opposite the greater angle.

\[\text{GIVEN: Triangle } \triangle ABC \text{ (Fig. 11)} \]
\[\text{in which } \angle A > \angle B.^{*} \]
\[\text{TO PROVE: } BC > CA. \]
\[\text{ANALYSIS: If } BC \text{ is greater than } CA, \text{ then by the preceding proposition angle } A \text{ must be greater than angle } B. \text{ But it was given at the outset that angle } A \text{ is greater than angle } B. \text{ At first thought, therefore, it might appear that we could simply reverse the steps here and have a perfectly good direct proof. But we have not yet shown that it is necessary for } BC \text{ to be greater than } CA \text{ in order that angle } A \text{ shall be greater than angle } B. \text{ For it might possibly be true that angle } A \text{ could be greater than angle } B \text{ when } BC \text{ was equal to } CA \text{ or even less than } CA. \text{ So we must consider these two cases also.} \]

\[^{*}\text{The symbol } < \text{ means less than, and the symbol } > \text{ means greater than. In each case the lesser quantity is written at the small (closed) end of the symbol, and the greater quantity is written at the large (open) end.}\]

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If $BC$ were equal to $CA$, then angle $A$ would have to be equal to angle $B$, but this is contrary to what was given.

If $BC$ were less than $CA$, then by the preceding proposition angle $A$ would have to be less than angle $B$; and this also is contrary to what was given.

But $BC$ must be greater than, equal to, or less than $CA$; and we have just shown that it cannot be either equal to or less than $CA$. The only possibility remaining is that $BC$ shall be greater than $CA$.

Sometimes, when we cannot make a definite and conclusive analysis of a proposition, we can analyze it in such a way that we exhaust all possible cases. All these partial analyses taken together constitute the proof, which cannot be stated directly. This is called the INDIRECT METHOD. It is perfectly sound logically, but frequently seems unsatisfactory because of its very indirectness. It is often called the method of reductio ad absurdum because by it every possibility except one is shown to be absurd and contrary to previously established propositions.

REFUTATION

In considering a question from real life for which we are able to give a direct proof, it is often wise to add to the direct proof a refutation of each contention of our opponent. It is not logically necessary to do so; but it adds to the force of our argument if we can show not only that we are right, but also that our opponent is wrong.

GEOMETRY AS AN IDEAL LOGICAL SYSTEM

In solving a problem from everyday life, we have no method that carries us inevitably to one answer, even though we present all the facts as logically as possible. Instead we are obliged to weigh one fact against another,
to consider all alternatives, and to choose the most likely conclusion. But even when all the facts and figures are marshaled in order, we have at best only a most probable answer, an answer that would be right in the majority of cases though wrong in the case of many individuals.

The main thing geometry gives us is the ideal of a logical system and of precise thinking, and some acquaintance with the language in which logical arguments are usually expressed. The answer to a problem in actual life can often be obtained by further investigation of the actual facts, while in geometry it can always be obtained by reasoning alone.
Maps of airplane routes illustrate the first assumption of our geometry: Number differences measure distances. In the map above, the number shown for each city is its distance in miles by plane from New York City.

Any figure can be reproduced either exactly or on any enlarged or diminished scale. The photographic enlarger pictured here makes use of this idea.

On the next page we shall begin a discussion of the basic assumptions underlying our geometry. Before we discuss these assumptions from a strictly logical point of view, let us pause for a moment to see how some of them relate to life about us, as suggested in the pictures on this page.

Surveyors make frequent use of another assumption of our geometry: Number differences measure angles.
The Five Fundamental Principles

We have learned in Chapter 1 what constitutes a good proof and have learned to distinguish between good and faulty reasoning. We have seen that we cannot prove propositions in geometry without first of all making a careful statement of just what we are assuming. We have seen the need also for certain definitions and undefined terms. These things we have learned from our consideration of only a few geometric propositions. We could have gone on to prove other propositions. But eventually we should have discovered that we needed more assumptions, including a few of a more general nature, to give us an adequate basis for geometry.

In this chapter, therefore, we start afresh and make a careful statement of the assumptions underlying our geometry. We shall need only five and shall refer to them usually as the five fundamental principles. These five assumptions are quite different from those we started with on page 20. Like those, these have to do with points, lines, distances, angles, and triangles. But, in addition, these new assumptions involve the laws of number* that underlie arithmetic and algebra.

*These laws of number are listed for convenience at the end of the book and are intended more for the teacher than for the student.
With the help of these five assumptions we shall prove as many geometric propositions as possible. Any such statement, once proved, we can use to help in proving still other propositions. Strictly, we shall be concerned only with propositions concerning plane figures, for this is a plane geometry that we are building. Consequently our propositions will apply only to figures in a plane surface unless specifically stated otherwise.*

Any definitions that we need will be given as the need for them arises. Certain terms, namely, number, order, equal, point, straight line, distance between two points, and angle between two lines, we shall take as undefined terms. We shall need also those undefined terms commonly employed in every sort of logical reasoning, as, for example, is, are, not, and, or, but, if, then, all, every. The word line will ordinarily be understood to mean a straight line. Part of our undefined notion of straight line and of plane is that each of these is a collection of points; also, that a straight line through any two points of a plane lies wholly within the plane. We shall assume also that a straight line divides a plane into two parts, though it is possible to prove this.

We were not obliged to assume this particular list of five fundamental principles. Many other lists are possible. Some of your friends may be studying geometry from books quite different from this one. In such books the proof of a certain proposition will look quite different from the proof given in this book. Their proof follows just as logically from their assumptions as our proof follows from our assumptions. The correctness of a proof or of any other course of reasoning cannot be judged independently of the assumptions on which it rests. We like our assumptions, because they are as simple and rea-

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*We shall not attempt to provide a rigorous foundation for the occasional references to solid geometry that we shall wish to make.
sonable as any others and make the proofs of the other propositions in geometry easier for the most part. Consequently we can prove easily many interesting propositions that others would call very hard.

In the following pages we shall discuss the five principles or assumptions, together with the related ideas that we take as the basis of our geometry. Read these discussions carefully but without feeling that you must study them laboriously. The language may seem strange and new to you, but the ideas will be familiar.

**PRINCIPLE 1**

Our first principle deals with points and lines and with the notion of distance between points. It is called the Principle of Line Measure and tells us in effect that we can measure the distance between two points by means of a scale, or ruler, just as we have always done. The ruler may have a scale of inches or centimeters or other units of length marked off on it. If we like, we may hold it upside down while measuring distances; it does not matter. All this implies that a line consists of a multitude of points each one of which can be tagged with a different number. We can say all this more briefly as follows.

**Principle 1. Line Measure.** The points on any straight line can be numbered so that number differences measure distances.

\[\begin{align*}
2.5 & \quad 3.5 & \quad 4 & \quad 5.25 \\
A & \quad B & \quad C & \quad D
\end{align*}\]

Fig. 1

You need not memorize this statement. You can refer to it simply as the Principle of Line Measure. The idea itself is obvious enough. For example, Fig. 1 shows four
points, A, B, C, and D, on a straight line. These four points and all the other points on the line may be numbered in order, and then we can say that the distance $AB$ is equal to the difference between the numbers corresponding to $A$ and $B$. In similar fashion we can measure the distance $BC$, the distance $AC$, and so forth. Instead of actually putting the numbers on the line, it is easier to lay a ruler, already marked, along the straight line. Instead of writing "the distance $BC$," hereafter we shall write simply $BC$ or $CB$. "The distance $BC$," "the length $BC$," "the distance $CB$," "the length $CB$," "$BC$," and "$CB$" shall all have the same meaning; the order of the letters has no significance.

**EXERCISES**

1. Find the distances $AB$, $BC$, $CD$, $AC$, $BD$, and $AD$ in Fig. 1 on page 40 and show numerically that:
   (c) $AB + BC = AC$
   (b) $AB + BC + CD = AD$
   (c) $AD = AB + BD = AC + CD$

2. (c) CD is how many times as long as $BC$?
   (b) If the numbers attached to the points A, B, C, D in Fig. 1 were changed to represent numbers of centimeters instead of inches, would $AB$ still be twice as long as $BC$?

3. How many degrees is it from the freezing-point of water to the boiling-point of water as measured on
   (a) the ordinary Fahrenheit thermometer?
   (b) the centigrade thermometer?

4. (c) Ruth is $4 \frac{1}{2}$ feet tall, and John is $5 \frac{1}{4}$ feet tall. John is how much taller than Ruth?
   (b) Ruth's height is what fractional part of John's height?
5. (a) Eleanor is 54 inches tall, and Henry is 63 inches tall. Henry is how much taller than Eleanor?

(b) Eleanor’s height is what fractional part of Henry’s height?

6. (a) Paul is 160 centimeters in height, and Virginia is 137 centimeters. Paul is how much taller than Virginia?

(b) Virginia’s height is what fractional part of Paul’s height?

Although it makes no difference what particular units we use, we must employ the same units throughout any piece of work.

Fig. 2 shows what happened when four different pupils attempted to measure the distance EG on a certain straight line. They all used the same scale to measure this distance. Did they all get the same result?

Fig. 2

The Principle of Line Measure (Principle 1) tells us that the distance EG is measured by the difference between the numbers corresponding to E and G. Strictly speaking, this difference has no sign. If we like, however, we can distinguish between the distance EG and "the directed distance EG," defining the latter as the number corresponding to G minus the number corresponding to E. This "directed distance EG," the symbol for which is $EG$. 
will be positive or negative: positive when the numbers increase (algebraically) as we go from $E$ to $G$, and negative when the numbers decrease (algebraically) as we go from $E$ to $G$. We see, moreover, that regardless of the way in which the line is numbered, $EG = -GE$.

Ordinarily, however, we shall not concern ourselves with such distinctions. The distance between two points, such as $E$ and $G$, shall be simply the numerical difference between the numbers corresponding to $E$ and $G$. That is, the distance $EG$ (Fig. 2) is 4.2 in each of the four cases illustrated; and, similarly, the distance $GE$ is also 4.2.

We can see from Principle 1 that if a point $Q$ on a line is numbered 7, then there are two distinct points on the line at a distance 2 from $Q$. These points correspond to the numbers $7-2$ and $7+2$. For every point $Q$ on a line there are two and only two distinct points on the line at a distance $d$ from $Q$. If $Q$ has the number $q$, these two points will be numbered $(q-d)$ and $(q+d)$.

NOTION OF BETWEENNESS FOR POINTS ON A LINE. As stated in Principle 1, the points on an endless straight line can be numbered so that number differences measure distances; there are various ways in which this can be done. Let us suppose that by one method of numbering the points on a straight line the number 3 corresponds to the point $A$ and the number 5.2 corresponds to the point $C$, as shown in Fig. 3. Then any point whose number lies between 3 and 5.2 will lie between $A$ and $C$ on the line.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3.8</td>
<td>5.2</td>
</tr>
</tbody>
</table>

Fig. 3

Any point $B$ lying on the line through points $A$ and $C$ is said to be between $A$ and $C$ if the numbers corresponding to $A$, $B$, and $C$ occur in order.
That part of the endless line through A and C which contains A and C and all the points between them is called the LINE SEGMENT AC.

If the point B corresponds to the number 3.8 (Fig. 3), then that part of the line which contains B and C and all the points whose numbers lie between 3.8 and 5.2 is called the line segment BC. The words "line segment" mean literally "a part cut out of the line." Strictly speaking, "the line AC" is the endless line through points A and C, and we should be careful to call that part of the line which lies between A and C "the line segment AC." But often, when it causes no confusion, we say "line AC" for short when we really mean "line segment AC."

If AB = BC, B is said to BISECT the line segment AC. B is also called the MID-POINT of the line segment AC.

PRINCIPLE 2

Our second principle also deals with points and lines. This principle, which requires no explanation, may be stated as follows.

Principle 2. There is one and only one straight line through two given points.

This line is to be thought of as endless, even though our diagrams show such lines as having two ends, as in Fig. 3 on page 43.

INTERSECTION OF Two LINES. By means of Principle 2 we can prove at once that two distinct straight lines cannot have more than one point in common. In order to prove this, let us see what would happen if the two straight lines had two points in common. There would then be two different straight lines through these two points, which is contrary to Principle 2. Therefore two
distinct straight lines cannot have two points in common, or three, or four, or any larger number.

When two lines have a point in common, we say that they intersect.

We have just proved that two straight lines intersect at one point or not at all. If they intersect, we call the point that they have in common their point of intersection.

**PRINCIPLE 3**

We now have established the Principle of Line Measure and have considered distances and lengths. Next we shall formulate the Principle of Angle Measure. Since our notion of angle is bound up in the notion of half-line, we shall first explain what we mean by a half-line.

If we select a point \( P \) anywhere on an endless straight line, we may think of this point as dividing the straight line into two half-lines each of which has \( P \) as an endpoint. Since (by Principle 1) the points of this endless straight line can be numbered so that number differences measure distances, we may define its two half-lines as follows: One half-line with end-point \( P \) is made up of the end-point \( P \) and all points whose numbers are greater than the number corresponding to \( P \); the other is made up of the end-point \( P \) and all points whose numbers are less than the number corresponding to \( P \). Each of these half-lines has but one end-point. Either half-line may be considered by itself alone, as indicated in Fig. 4.
If two half-lines have the same end-point (Fig. 5), they are said to form two angles. When we speak of the angle between $VA$ and $VB$, we ordinarily think of the lesser of these two angles. This angle is usually referred to as the angle $AVB$, in which the middle letter always represents the end-point which the two half-lines have in common. This point $V$ is called the vertex of the angle. The half-lines $VA$ and $VB$ are called the sides of the angle. $A$ and $B$ serve simply to distinguish the two sides of the angle; they represent any point (other than $V$) on their respective half-lines. We shall not make any distinction between angle $AVB$ and angle $BVA$; both expressions will ordinarily denote the lesser angle formed by the half-lines $VA$ and $VB$.

When two half-lines with a common end-point $V$ together make an endless straight line, as in Fig. 6, each angle formed by the two half-lines is called a straight angle.

Our third principle concerns the measurement of angles. It says in effect that angles can be measured by means of a protractor in the way that we have learned to measure them. If, for example, we should take a wheel of twelve spokes and mark the numbers 30, 60, 90, 120, . . . 360 in order on the rim, putting one number at the end of each spoke, we could say that the measure of the angle between the 90 spoke and the 150 spoke would be 60. This would still be true if we had used the numbers 31, 61, 91, 121, . . . 361, or if we had used another scale which did not suggest the customary unit for the measurement of angles, the degree. Now we shall say all this more briefly.

*This word "lesser" has no meaning from a strictly logical point of view until we have discussed angle measure. A valid distinction between the two angles that are formed by two half-lines having the same end-point can be made with the help of the idea of "betweenness."*
**Principle 3.** ANGLE MEASURE. All half-lines having the same end-point can be numbered so that number differences measure angles.

For example, Fig. 7 shows five half-lines having the end-point \( O \) in common. These five half-lines and all the other half-lines that have \( O \) as their end-point can be numbered in order, and then we can say that the measure of the angle between any two of these half-lines is equal to the difference between their numbers.\(^*\) For convenience in reading we shall arrange these numbers around a circle drawn with \( O \) as its center and shall letter the points where the circle cuts the half-lines. It must be emphasized, however, that the numbering of the half-lines is independent of any circle. Logically, the circle may be dispensed with; it is introduced only for convenience. The angle between the half-lines \( OA \) and \( OB \) has the measure 40 \(-\) 10, or 30. The angle between the half-lines \( OB \) and \( OC \), usually read \( "\text{the angle } BOC,\)" has the measure 50 \(-\) 40, or 10. What is the measure of angle \( COD? \) Of angle \( AOC? \)

\( "\text{The angle } AOB,\)" \( "\text{the angle } BOA,\)" \( "\text{LAOB,}\)" and \( \text{L BOA}\) shall all have the same meaning; the order of the letters has no significance.

\(^*\)This idea that all the half-lines having the same end-point can be numbered should not be confused with the idea given in the definition of a half-line, namely, that all the points on a half-line can be numbered. The spokes of a wheel may be numbered in turn, and then the points on each individual spoke may also be numbered.

We shall use the word "angle" to denote not only a geometric figure formed by two half-lines having the same end-point, but also the measure of that figure. Though ambiguous, this will cause no confusion.
Since the measure of angle $AOC$ (Fig. 7) equals the measure of angle $AOB$ plus the measure of angle $BOC$, we can say that angle $AOC$ equals angle $AOB$ plus angle $BOC$; or, more briefly, $\angle AOC = \angle AOB + \angle BOC$. And since the measure of angle $AOC$ equals four times the measure of angle $BOC$, we can say that $\angle AOC = 4\angle BOC$.

If $\angle AOB = \angle BOC$, half-line $OB$ is said to bisect the angle $AOC$, and $OB$ is the bisector of angle $AOC$.

**EXERCISES**

In Fig. 7 on page 47 show that

1. $\angle AOD = \angle AOB + \angle BOC + \angle COD$.
2. $\angle AOD = \angle AOC - \angle BOC + \angle BOD$.
3. $\angle BOE = \angle AOC + \angle BOD + \angle COE - \angle AOB - \angle BOC - \angle COD$.

A few of the endless variety of ways in which half-lines can be numbered so that number differences measure angles are shown in Figs. 8-11. Half-lines are customarily numbered counter-clockwise with positive numbers. This is merely for convenience and is not absolutely necessary; for we could just as well agree to number the same half-lines clockwise with positive numbers, or, as shown in Fig. 9, clockwise with negative numbers.
Ordinarily we shall have no need to continue our numbering after completing the circuit once. If we like, however, we may continue our numbering indefinitely; angle $KOL$ (Fig. 9) will still have the measure 30, whether it be regarded as $150 - 120$, or $510 - 480$, or $(-210) - (-240)$. We could indeed consider the measure of angle $KOL$ to be $510 - 120$, or $30 + 360$; or even $30 + n \cdot 360$ where $n = 0, 1, 2, 3, \ldots$ or $-1, -2, -3, \ldots$. It is not easy to do this and avoid confusion, however. To be strictly accurate, we ought to introduce the idea of the "directed angle $KOL$," defining it as "the number of the half-line $OL$ minus the number of the half-line $OK$." We ought, moreover, to indicate that when this difference is a positive number, the directed angle $KOL$ is to be considered counter-clockwise; and when negative, the directed angle $KOL$ is to be considered clockwise. The measure of each of the corresponding directed angles $LOK$ would be the negative of each of the foregoing, respectively. Usually we shall not concern ourselves with such distinctions but shall take as the measure of the angle between two half-lines the smallest numerical difference of the numbers of the half-lines.

**PRINCIPLE 4**

The commonest method of numbering half-lines so that number differences measure angles is the method shown in Fig. 8 on page 48, where $\theta$ and 360 serve to number the same half-line. In this case the *unit of angle*
measure is called a DEGREE. A precise definition of this unit of angle measure, which is written 1°, is that it is the measure of \(\frac{1}{360}\) of a straight angle. Strictly, however, we have no right to use this definition without first assuring ourselves that all straight angles have the same measure. We could prove this as a theorem if we should introduce Principle 5 at this point to help us; but the proof is somewhat long and a bit unusual in character. Consequently, instead of proving it as a theorem, we shall take it as an assumption. So we have

**Principle 4.** All straight angles have the same measure.

In using a degree as the unit of angle measure we observe that an angle may be considered to have many measures that differ by multiples of 360°. Ordinarily, however, we shall consider the measure of a lettered angle, such as \(\angle LKOL\) in Fig. 9 on page 48, to be less than 360°.

An angle of 90 degrees is called a RIGHT ANGLE. Angles of less than 90 degrees are called ACUTE ANGLES, and angles of more than 90 degrees and less than 180 degrees are called OBTUSE ANGLES.

If two lines meet at a point 0 so that the angle between two of their half-lines has the measure 90°, the lines are said to be PERPENDICULAR. If, for example, the lines \(LL'\) and \(MM'\) in Fig. 12 meet at 0 so that angle \(\angle LOM = 90°\), the lines \(LL'\) and \(MM'\) are said to be perpendicular. Moreover, the angles \(\angle NOL', \angle LOM',\) and \(\angle MOL\) will also be right angles. For since \(LL'\) is a straight line, angle \(\angle LOL'\) will be a straight angle, or 180°. The half-line \(OL'\) will in this case be numbered 180, and angle \(\angle MOL'\) will have the measure 180 - 90, or 90°.
EXERCISES

1. Show that angles $L'OM'$ and $M'OL$ in Fig. 12 are also right angles.

2. Through how many degrees does the minute-hand of a clock turn in going from "ten after" the hour to "twenty-five after"? From "ten after" to "twenty after"? From 6:05 to 6:15? From 6:15 to 7:05? From 6:05 to 7:05?

3. Through how many degrees does the minute-hand of a clock turn in going from 8:10 to 9:20? From 10:50 to 2:35? From 3:13 to 3:51?

4. What time will it be when the minute-hand has turned 72 degrees from its position at 8:19 o'clock?

5. What time will it be when the minute-hand has turned 726 degrees from its position at 11:47 P. M.?

6. If in Fig. 5 on page 46 the lesser angle $AVB$ is $117^\circ$, how large is the greater angle $AVB$?

7. If two endless straight lines $AB$ and $CD$ intersect at $V$, they may be considered as forming four half-lines. See Fig. 13.

8. If in Fig. 13 half-line $VA$ is numbered 45, how will half-line $VB$ be numbered? (Remember that we are measuring angles in degrees.)

9. In general, if one half-line $VA$ of an endless straight line $AB$ is numbered $r$, how will the other half-line $VB$ be numbered?

10. If in Fig. 13 half-line $VA$ is numbered 0, 360, and $LAVC=132^\circ$, how will half-lines $VC$, $VB$, and $VD$ be numbered?
10. If in Fig. 13 half-line VA is numbered $a$, $a+360$, and $LAVC=\pi$, how will half-lines VC, VB, and VF be numbered?

11. If in Fig. 13 $LAVC=125.8^\circ$, how large is $LCVB$? $LBVD$? $LDVA$? What is the sum of these four angles?

12. If in Fig. 13 $LAVC=90^\circ$, how large is $LCVB$? $LBVD$? $LDVA$? What is the sum of these four angles?

The angles AVC and BVD in Fig. 13 are sometimes called vertical angles.

13. Name another pair of vertical angles in Fig. 13. From Ex. 10 and 12 it follows that vertical angles are equal.

14. If in Fig. 13 $LAVC$ were equal to $LCVB$, how large would each angle be?

*15. Show that the lines bisecting angles AVC and eVB (Fig. 13) are perpendicular.

16. If in Fig. 13 the lesser angle AVC has the measure $90^\circ$, how large is the greater angle AVC?

In the above exercises we have been using the degree as the unit of angle measure; we defined it as the measure of an angle that is $\frac{\pi}{180}$ of a straight angle. Why should the measure 180 degrees be assigned to a straight angle, rather than the measure 5, 6, or 50? The answer is to be found in the history of our civilization. It seemed natural to the early inhabitants of Mesopotamia that they should count on their fingers. They then gradually formed the habit of grouping things by tens. They encountered many difficulties in dealing with fractions, however, and were obliged to resort to fractions that had easily divisible denominators. In order to express fractional parts, therefore, they employed twelfths and

*To THE TEACHER: The exercises are important for their content and for the development of the subject. None should be omitted without careful consideration. Those marked with a star are of especial importance.
sixtieths, rather than tenths. Our present practice reveals traces of this in our foot of 12 inches, our Troy pound of 12 ounces, our day of 24 hours, our hour of 60 minutes, and our minute of 60 seconds. It required centuries before the Hindu-Arabic numerals became established, and it was many more centuries before the decimal point and decimal fractions were invented. The invention of decimal fractions made it no longer necessary or desirable to employ units subdivided into twelfths and sixtieths; in fact, the decimal system lends itself better to the use of units of measure that are divided into tenths, hundredths, thousandths, and so forth, as exemplified by the metric system. It was this idea that led the French to divide the straight angle into 200 equal parts, called grades (pronounced grahd), so that their measure of a right angle is 100 grades. See Fig. 10, page 49. Artillerymen often use a unit of angle measure called the mil, such that a straight angle has 3200 mils. See Fig. 11, page 49.

Throughout any piece of work, of course, we must use the same units of angle measure.

**EXERCISES**

Express each of the following in two other systems of angle measure. (90 degrees = 100 grades = 1600 mils)

1. 30°
2. 135°
3. 25 grades
4. 60 grades
5. 100 mils

**NOTION OF BETWEENNESS FOR HALF-LINES WITH COMMON END-POINT.** As we have just learned, there are various methods of numbering half-lines in angle measure. Let us suppose that in one method of numbering the

![Fig. 14](image)
number 41 corresponds to the half-line $OA$ and the number 78 corresponds to the half-line $OC$, as shown in Fig. 14 on page 53. Then any half-line whose number lies between 41 and 78 will lie between $OA$ and $OC$. This is just like the notion of betweenness with respect to points on a straight line, as explained on page 43.

There can be only two half-lines $OB$ and $OB'$ that make with the half-line $OA$ an angle equal to 30°. See Fig. 15. This follows immediately from the Principle of Angle Measure (Principle 3) by a course of reasoning like that used in the second paragraph on page 43. For while there are infinitely many ways of numbering half-lines so that number differences measure angles, these infinitely many ways of numbering differ among themselves in only three respects: (1) as to the unit used; (2) as to the beginning point of the scale; (3) as to whether the numbering shall be counter-clockwise or clockwise. In this case we have specified that the unit to be used in numbering shall be the degree; and since we are concerned only with the differences between numbers, the beginning point of the scale is of no consequence. So, whether the numbering be counter-clockwise or clockwise, we get the same two half-lines, $OB$ and $OB'$.

Similarly, for all values of $n$ between 0 and 180, there can be only two half-lines, one on each side of $OA$, that make with $OA$ an angle of $n$ degrees. Thus there are only two half-lines, one on each side of $OA$, that make with $OA$ an angle of 90 degrees. Therefore there can be but one line perpendicular to the line $AB$ at the point $O$ (Fig. 16).
Thus far the assumptions of this geometry have dealt with only the most elementary ideas concerning points, lines, and angles. In order to develop a complete system of geometry, we must be able to deal with geometric figures that involve points, lines, and angles in a somewhat more complicated manner. For this purpose we shall need to define a few of the simplest of these more complicated arrangements of points, lines, and angles. We shall also need to assume a basic principle concerning them.

First we shall define the terms that we shall need. In Fig. 17 we say that the line segments \( AB, BC, CD, \ldots \) joining the points \( A, B, C, D, \ldots \) form the broken line \( ABCD \). We give the name \textit{polygon} to a broken line in which the first and last points are the same. See Fig. 18. We call the points \( A, B, C, D, \ldots \) the vertices of the polygon; we call the line segments \( AB, BC, CD, \ldots \) the sides of the polygon; and we call the interior angles \( ABC, BCD, CDE, \ldots \) the angles of the polygon.

Almost all the polygons that we shall meet in our work will have every angle less than 180°. Such polygons are sometimes called convex polygons in order to distinguish them from the unusual type that is shown in Fig. 23 on page 60.
We call a three-sided polygon a triangle and a four-sided polygon a quadrilateral. How many sides has a pentagon? A hexagon? A heptagon? An octagon? A decagon?

**EXERCISES**

1. Construct with ruler and protractor the polygon ABCDEA, given \( AB = 2 \) in., \( LABC = 131^\circ \), \( BC = 2\frac{1}{2} \) in., \( LBCD = 94^\circ \), \( CD = 1\frac{1}{4} \) in., \( LCDE = 116^\circ \), \( DE = 3 \) in.

2. Measure \( EA \) in the polygon of Ex. 1. Measure also angles \( DEA \) and \( EAB \) and find the sum of the five angles of the polygon. (This sum should be 5400.)

3. Using a ruler marked in centimeters and a protractor, construct the triangle \( ABC \), given \( AB = 7.5 \) cm., \( LABC = 70^\circ \), and \( BC = 8.2 \) cm. Find the length of \( CA \) and the number of degrees in each angle of the triangle. How many degrees are there in all three angles together?

4. A traveler goes 4 miles due east, then north 10° east (that is, 10° east of north) for 5 miles, then N 74° E for 5 miles, then S 12° E for 4\( \frac{1}{2} \) miles, then SW for 2 miles, then E 8° S for 3 miles, then N 40° E for 3\( \frac{1}{2} \) miles. How far is he from his starting-point? (Make a drawing to scale to find the answer.)

5. A fishing schooner is bound for Gloucester, which is 20 miles away and in the direction N 12° W from the schooner. The captain cannot head his vessel in that direction, however, because the wind is against him. So he sails N 10° E for 15 miles; then turns and heads W 10° S. How far should he continue on this second course so that when he "comes about" again and sails N 10° E he will be heading direct for Gloucester? How far will he have sailed in all? (You will need to make an exact drawing to find the answer.)
Now we need to define similar figures because the fifth principle that we shall take for granted involves such figures. Two polygons that have their corresponding angles equal and their corresponding sides proportional are called similar polygons.

Two geometric figures are similar if all corresponding angles are equal and all corresponding distances are proportional. The order of the angles and sides in one figure may be either the same as, or the reverse of, the order of the corresponding angles and sides in the other figure, as shown in Fig. 19. Notice that corresponding vertices (corners) in all three of the similar triangles in this figure are marked with the same letter. Are the vertices lettered in counter-clockwise order in all three triangles? Which two triangles have their corresponding vertices in the same order? Are the corresponding vertices in the other triangle in the reverse order?

When we say that all corresponding distances in polygons $ABCDE$ and $A'B'C'D'E'$ are proportional, we mean that every side and every diagonal in polygon $ABCDE$ is, say, 3 times as long as the corresponding side or diagonal in polygon $A'B'C'D'E'$. We call the 3 the factor of proportionality. It tells us how the distances in polygon $ABCDE$ compare with those in polygon $A'B'C'D'E'$. 
We can express this comparison as a proportion in two ways. We may write $\frac{AB}{A'B'} = \frac{3}{1}$, $\frac{BC}{B'C'} = \frac{3}{1}$, $\frac{AD}{A'D'} = \frac{3}{1}$, or we may write $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AD}{A'D'}$. Notice that the second way merely says that the corresponding distances are in proportion; it does not show the factor of proportionality, 3. In order to include the idea that $AB$ is 3 times $A'B'$, we must indicate that one of these fractions is equal to 3.

The two proportions shown above may be used to state the relation between the corresponding distances of any two similar polygons if instead of using 3 as the factor of proportionality we use $k$, where $k$ may be any number.* We write, then, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AD}{A'D'} = k$.

The fifth and last principle that we shall take for granted involves similar triangles. Since Principle 5 deals with the first of three closely related situations of this sort, we shall refer to it as Case 1 of Similarity. We shall consider Cases 2 and 3 in the next chapter.

**Principle 5.** CASE 1 OF SIMILARITY. Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.

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*This number $k$ can be any real number, rational or irrational.
In triangles $ABC$ and $A'B'C'$ in Fig. 20, if $\angle A'B'C' = \angle ABC$, $A'B' = k \cdot AB$, and $B'C' = k \cdot BC$,
then $C'A' = k \cdot CA$, $\angle LB'C'A' = \angle BCA$, and $\angle LC'A'B' = \angle CAB$.

It follows, therefore, that in assuming Case 1 of Similarity we are assuming, in effect, that a given triangle can be reproduced anywhere, either exactly or on any enlarged or diminished scale. The same is true of polygons; for a polygon can always be thought of as being composed of a certain number of triangles. See Fig. 21.

![Fig. 21](image)

We shall find Principle 5 (Case 1 of Similarity) very helpful when we wish to prove that certain angles are equal or that a certain line segment is some multiple of another. When the factor of proportionality is 1, we can use Principle 5 to prove that two line segments are equal.

Two similar triangles in which $k$ is 1 are called equal triangles because their corresponding sides and angles are equal. Other books on geometry often refer to equal triangles as "congruent" triangles. They do this to indicate not only that corresponding sides and angles are equal, but also that this equality can be shown by moving one triangle and fitting it on the other. They define
"congruent" in terms of the undefined ideas of "move" and "fit." The logical foundation of our geometry is independent of any idea of motion.*

**EXERCISES**

1. In triangle $ABC$, $AB=6$, $BC=8$, and $LABC=62^\circ$. In triangle $A'B'C'$, $A'B'=9$, $B'C'=12$, and $LA'B'C'=62^\circ$. $C'A'$ is how many times as long as $CA$?

2. In triangles $ABC$ and $A'B'C'$, $A'B'=3AB$, $B'C'=3BC$, and $LA'B'C'=LABC$. What can you say about $C'A'$ and $CA$? About $LB'C'A'$?

3. By means of a ruler, marked or graduated either in inches or centimeters, and a protractor, enlarge the triangle $ABC$ (Fig. 22) in the ratio 3 to 2. First draw $A'B'=\frac{3}{2}AB$; then make $LA'B'C'=LABC$, and draw $B'C'=\frac{3}{2}BC$.

![Fig. 22](image1)

![Fig. 23](image2)

Hereafter we shall usually call a marked ruler—however it may be graduated—a *scale*. In particular we may refer to an *inch-scale* or a *centimeter-scale*. A plain ruler without graduations we shall call an *unmarked ruler*, or a *straightedge*.

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*Later, when we wish to link our geometry with problems of the physical world about us, we shall simply take as undefined the idea of motion of figures (without change of shape or size). It is possible, however, to develop the idea of motion in terms of number. See Chapter 8, pages 228-231, and Chapter 9, page 242.
4. Reproduce the polygon in Fig. 23 on page 60 by means of a drawing so that it will be enlarged in the ratio 5 to 4.

5. Draw a triangle on any convenient sphere (a tennis ball will do).* Then extend two of the sides until each of these sides is double its original length, as shown in Fig. 24. What can you say about the third side and the other two angles? Do this for several triangles.

6. Principle 5 is not true for triangles on a spherical surface. Can Principles 1 and 3 be so modified as to apply to figures on a sphere?

*7. Prove that if two convex quadrilaterals are similar they can be divided into triangles so that corresponding triangles will be similar. That is, show that a pair of corresponding triangles has an angle of one equal to an angle of the other and that the sides including these angles are proportional. See Fig. 25.

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*The sides of this triangle will be arcs of circles whose centers are at the center of the sphere. Such circles on a sphere are called "great circles" to distinguish them from "small circles." By looking at a globe showing the map of the world, you can easily see that the Equator and the meridians are great circles, while the Arctic Circle and the Tropic of Cancer are small circles.

The angle between two sides of a spherical triangle is the same as the angle between tangents drawn to these sides at the point where they intersect to form the vertex.
8. Prove the theorem stated in Ex. 7 if the quadrilaterals are not convex, as in Fig. 26. Does it make any difference whether we use diagonals $EG, E'G'$ or diagonals $FH, F'H'$?

*9. Prove that if two convex quadrilaterals can be divided into triangles so that corresponding triangles are similar the quadrilaterals will be similar.

10. Prove the theorem stated in Ex. 9 if the quadrilaterals are not convex.

*11. Prove that if two polygons are similar they can be divided into triangles so that corresponding triangles will be similar. Suggestion: What must we know in order to prove triangles $ACD$ and $A'C'D'$ in Fig. 27 similar? Can we prove $A'C'=k \cdot AC$ and $C'D'=k \cdot CD$? Can we prove that $L A'C'D'=LACD$?

*12. Prove that if two polygons can be divided into triangles so that corresponding triangles are similar the polygons will be similar.
13. Notice how points have been selected along the curved line showing a man’s head in Fig. 28. The points have been connected by straight lines to form a broken line. Enlarge this broken line in the ratio 2 to 1, using very light pencil lines. Then sketch the enlarged profile of the head in ink. If you prefer, substitute a picture from a magazine, copying the angles by pricking through the points with a pin. Using a protractor on a small drawing is very awkward.

14. Enlarge another picture in the ratio 3 to 2.

15. Reduce another picture in the ratio 1 to 2.

*16. In Fig. 29 prove that any point $P$ in the line perpendicular to $AB$ at its mid-point $M$ is equally distant from $A$ and $B$. What else can you prove to be true about the figure? $PM$ is called the PERPENDICULAR BISECTOR of $AB$.

*17. In triangle $ABC$ lines are drawn joining the mid-points of the three sides. Prove that three of the four small triangles so formed are similar to the triangle $ABC$.

18. What would you need to know in order to prove the fourth small triangle similar to triangle $ABC$?

19. In Fig. 30 find the distance between the points $A$ and $B$ separated by a pond without actually measuring the distance $AB$. Prove that your method is correct.
20. In Fig. 31, \( AB' = 1.5AB \) and \( AC' = 1.5AC \). Find the ratio \( \frac{B'C'}{BC'} \).

To find the ratio \( \frac{BB'}{AB} \) (Fig. 31), we need only observe that \( BB' = AB' - AB \); and hence \( \frac{BB'}{AB} = \frac{AB'}{AB} - 1 \), or in this case \( \frac{2}{3} - 1 \), or \( \frac{1}{3} \). This is a general method of great importance; we shall frequently find it useful.

21. If in Fig. 31, \( \frac{AB'}{AB} = \frac{5}{3} \), find \( \frac{BB'}{AB} \).

22. If in Fig. 31, \( \frac{AB'}{AB} = 1.2 \), find \( \frac{BB'}{AB} \).

23. Given \( \frac{AB}{AB'} = \frac{m}{n} \), find \( \frac{BB'}{AB} \).

24. Find the ratio \( \frac{B'C'}{BC} \) in Fig. 31, remembering that Ex. 20 above gives important information about the figure.

25. Find the ratio \( \frac{B'C'}{BC} \) (Fig. 31).

26. Prove that \( \frac{BB'}{AB} = \frac{AC'}{BC} \) (Fig. 31).

27. Given \( \frac{BB'}{AB} = \frac{1}{2} \) (Fig. 31), how could we find the ratio \( \frac{AB}{AB'} \) (Note that \( AB' = BB' + AB \)).

28. If in Fig. 31, \( \frac{BB'}{AB} = \frac{2}{3} \), find \( \frac{AB}{AB'} \).

29. If in Fig. 31, \( \frac{BB'}{AB} = \frac{1}{5} \), find \( \frac{AB}{AB'} \).

30. Given \( \frac{BB'}{AB} = \frac{n}{m} \), find the ratio \( \frac{AB}{AB'} \).
If in Fig. 32, $MN$ divides $PQ$ and $PR$ so that $\frac{a}{b} = \frac{c}{d}$, it follows at once that $\frac{b}{a} = \frac{d}{c}$; for reciprocals of equal numbers are equal. Prove this by multiplying each side of the given equation by $bd$ and then dividing each side of the resulting equation by an appropriate number.

If four numbers are in proportion so that the first is to the second as the third is to the fourth, then the second is to the first as the fourth is to the third.

If four numbers are in proportion so that the first is to the second as the third is to the fourth, then the first is to the third as the second is to the fourth. Suggestion: Add 1 to each side of the given equation.

If four numbers are in proportion so that the first is to the second as the third is to the fourth, then the sum of the first and second is to the first (second) as the sum of the third and fourth is to the third (fourth). Also, if a line $MN$ divides the sides $PQ$ and $PR$ of triangle $PQR$ (Fig. 32) so that the corresponding segments of each side are in proportion, then each side is to either one of its segments as the other side is to the corresponding segment.
If in Fig. 32, \( PM = PN \), prove that \( \frac{MQ}{PM} = \frac{NR}{PN} \).

That is, given \( \frac{a+b}{c} = \frac{d}{e} \), prove that \( \frac{b}{d} \). 

Suggestion: Subtract 1 from each side of the given equation.

If in Fig. 32, \( MQ = NR' \), prove that \( \frac{PM}{MQ} = \frac{PN}{NR'} \).

That is, given \( \frac{e}{d} = \frac{f}{d} \), prove that \( \frac{e-b}{f} = \frac{f-d}{d} \).

If four numbers are in proportion so that the first is to the second as the third is to the fourth, then the first minus the second is to the second (first) as the third minus the fourth is to the fourth (third). Also, if a line \( MN \) divides the sides \( PQ \) and \( PR \) of triangle \( PQR \) (Fig. 32) so that each side is to either one of its segments as the other side is to the corresponding segment, then the corresponding segments are in proportion.

When we say that \( MN \) divides two sides, \( PQ \) and \( PR \), of triangle \( PQR \) proportionally, we mean not only that the corresponding segments are in proportion, but that both sides and either pair of their corresponding segments are in proportion.

\[ \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \]

Let \( k = \frac{a}{c} = \frac{e}{f} \). and write \( a=kb, c=kd, e=kf \).

In a series of equal fractions constituting a continued proportion, the sum of two or more numerators is to the sum of the corresponding denominators as any numerator is to the corresponding denominator.

The **perimeter** of a polygon is the sum of all its sides.

\[ \text{ } \]

*38. Prove that the perimeters of two similar polygons have the same ratio as any two corresponding sides.
Let us state again the principles and the related ideas explained in this chapter and list the terms defined.

1. **Line Measure.** The points on any straight line can be numbered so that number differences measure distances.

   The distance $AB$ we write as $AB$ or $BA$; $AB = BA$.

   The directed distance $AB$, or simply $AB$, is the number corresponding to $B$ minus the number corresponding to $A$; $\hat{AB} = -BA$.

**DEFINITIONS:** *between, line segment, bisect, mid-point*

2. There is one and only one straight line through two given points.

   Two distinct straight lines cannot have more than one point in common.

**DEFINITIONS:** *intersection of two lines, half-line, vertex (of angle), side (of angle), straight angle*

3. **Angle Measure.** All half-lines having the same end-point can be numbered so that number differences measure angles.

   The angle between $AO$ and $BO$ we can write as $LAOB$ or as $LBOA$; $LAOB = LBOA$.

**DEFINITION:** *bisector of angle*

   The directed angle $AOB$ is the number corresponding to $OB$ minus the number corresponding to $OA$.

**DEFINITION:** *degree*

4. All straight angles have the same measure (180°).

**DEFINITIONS:** *right angle, acute angle, obtuse angle, perpendicular, vertical angles, broken line, polygon, similar figures, factor of proportionality.*

"Proportion" is used without definition, its meaning being assumed from arithmetic.
5. **Case 1 of Similarity.** Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.

A given triangle can be reproduced anywhere, either exactly or on any enlarged or diminished scale.

**DEFINITIONS:** equal triangles, diagonal (of a polygon), perpendicular bisector, perimeter (of a polygon)

**EXERCISES**

1. In Fig. 33 what is the distance $AB$? The distance $BA$?

Ex. 2-8 refer to Fig. 33.

2. What is the directed distance $BA$?

3. What is $AB$?

4. Show that $AC + CB = AB$.

5. Show that $CB + BA = CA$.

6. Show that $CA + AB = CB$.

7. What is line segment $BC$?

8. What is line segment $BA$?

9. In Fig. 34 what is the angle $AOB$? What is the angle $BOA$?

Ex. 10-12 refer to Fig. 34.

10. What are the directed angles $BOC$, $COB$, $AOC$, and $COA$?

11. Show that directed $\angle BOC +$ directed $\angle COA =$ directed $\angle BOA$.

12. Number a half-line through 0 perpendicular to $OA$. 68
13. A man is at Sylvania, a tiny hamlet in a dense forest 9 miles S. E. of Toggenburg. From Sylvania he follows a road running N 12° E for 5 miles, then turns and goes due west for 6 miles. How far is he now from Toggenburg, and in what direction?

14. Reproduce a map of Ohio in the ratio 2 to 1 and show the principal cities.

15. In Fig. 35 $AB' = \frac{1}{2} AB$ and $AC' = \frac{1}{4} AC$. Find the ratios $\frac{BB'}{AB}$ and $\frac{BC''}{BC}$.
Mr. Farrell has just surveyed a plot of land for Mr. Krueger and is now in his office using his field notes to make a map of the plot. Since each angle measured by his transit is likely to be slightly in error, he is checking his field notes and distributing the errors as best he can among the several angles. His check depends upon the theorem that states that the sum of the three angles of a triangle is 180 degrees.

In the picture below, one man is holding the tape-line at a point 15 feet from the corner. The other man is holding it at a point 20 feet from the corner. If these points are 25 feet apart, the corner is a right angle. This is an application of the converse of the Pythagorean Theorem. The Pythagorean Theorem is: In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
CHAPTER 3

The Seven Basic Theorems

By means of the five assumptions of the preceding chapter, Principles 1-5, we shall proceed to prove seven fundamental propositions, or theorems, which we shall call Principles 6-12. Taken together, these twelve principles will enable us to prove all subsequent propositions in geometry.

Although we can prove all seven of these new principles, we shall prefer to take four of them for granted without proof at this time. These are Principles 6, 7, 8, and 11. Most people would grant immediately the truth of these four principles, and it is always irksome to go through the form of proving what seems to be obvious. At the end of our study of demonstrative geometry, when we have had much more experience in proving theorems, we shall see why it is interesting to discover how few assumptions we really need as a basis for our geometry. We shall not object then to proving what seems to be obvious. So although the proofs of these four principles are given in this chapter in their proper places, we shall skip them at this time, returning to them when we get to Chapter 10. By taking these four principles as assumptions for the time being, we can prove the propositions in the exer-
cises in Chapter 3 and can prove also Principles 9, 10, and 12.

**PRINCIPLE 6**

Principle 6 is called Case 2 of Similarity. Compare it with Principle 5 to see how Case 2 of Similarity differs from Case 1. Omit the analysis and proof of Principle 6 for the present. You will be asked to consider them later.

*Principle 6. Case 2 of Similarity.* Two triangles are similar if two angles of one are equal to two angles of the other.

![Diagram](Fig. 1)

**Given:** Triangles $ABC$ and $A'B'C'$ (Fig. 1) in which $LA = LA'$ and $LB = LB'$.

**To Prove:** Triangle $A'B'C'$ similar to triangle $ABC$.

**Analysis:** It is evident that we must in some way make the proof of this theorem depend on Case 1 of Similarity (Principle 5). To apply this principle directly, we should need to know the relations between sides including corresponding angles; but we are told nothing about the sides of these two triangles, and we can see that Principles 1, 2, 3, and 4 offer nothing helpful on this question. Principle 5 at least has to do with triangles and tells us one way of knowing when two triangles are similar.
We cannot apply this principle directly, but perhaps we can apply it indirectly. Let us try.

**PROOF:** Since nothing at all has been said about the sides of either triangle, we can assume that $A'B'=k\cdot AB$, where $k$ may be any positive number whatsoever.

Lay off on $B'C'$ (extended beyond the point $C'$ if necessary) the length $B'C''$ equal to $k\cdot BC$; we do not know where $C''$ will fall with respect to $C'$. By Case 1 of Similarity, the triangles $ABC$ and $A'B'C''$ are similar, and $\angle BAC=\angle B'A'C''$. But $\angle BAC=\angle B'A'C'$ (Given). Therefore $\angle B'A'C''=\angle B'A'C'$, and $C''$ must lie on $A'C'$ as well as on $B'C'$. Since these two lines can intersect in not more than one point, it follows that $C''$ must coincide with $C'$.

Therefore $B'C'=k\cdot BC$, and triangles $ABC$ and $A'B'C'$ are similar.

In other words, if in triangles $ABC$ and $A'B'C'$, $\angle A'=\angle A$, $\angle B'=\angle B$, and $A'B'=k\cdot AB$, then $B'C'=k\cdot BC$, $C'A'=k\cdot CA$, and $\angle L'C'=\angle LC$.

**EXERCISES**

1. If two triangles have two angles of the one equal to two angles of the other, the third angles are equal also. Why?

2. In triangle $ABC$ (Fig. 2), $AB = 6$, $\angle A = 40^\circ$, and $\angle B = 100^\circ$. In triangle $A'B'C'$, $A'B' = 9$, $\angle A' = 40^\circ$, and $\angle B' = 100^\circ$. How long is $A'C'$ compared with $AC$? How big is $\angle C'$ compared with $\angle C$?

3. In triangles $KLM$ and $K'L'M'$, $LK = LK'$, $\angle L = \angle M$, and $KM = 1.3 \times K'M'$. Compare $KL$ and $K'L'$; $LM$ and $L'M'$.  

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**Fig. 2**
4. Using a ruler marked in inches and a protractor, enlarge the triangle $ABC$ (Fig. 2 on page 73) in the ratio 5 to 4. First draw $A'C'$ equal to $\frac{5}{4}AC$; then make $LA'=LA$ and make $LC'=LC$.

5. What is the length of the unmarked line in Fig. 3?

6. If a 10-foot vertical pole casts a 6-foot horizontal shadow, how high is a tree that at the same time casts a 15-foot shadow?

7. Fig. 4 shows a 24-foot ladder leaning against a house with the foot of the ladder 5 feet from the house. How far from the side of the house is the rung that is 7 feet from the foot of the ladder?

8. Fig. 5 shows how a water pipe is being laid under the foundation of a building. The trench for the pipe meets one wall of the building at an angle of 25° and 10 feet in from the corner. How would you locate the point $B$ where the pipe emerges? Prove that you are right. Would your method work if the angle at corner $C$ were not a right angle?


*The line from a vertex of a triangle perpendicular to the opposite side is called an altitude of the triangle.* We also use the word "altitude" to mean the length of such a line, but we shall do this only when there is no likelihood of confusion.

*We shall assume, in anticipation of Principle 11, that there is only one such perpendicular from each vertex.*
*10. Prove that in triangle \( ABC \) (Fig. 6), \( bh = ck \), where \( h \) and \( k \) are the altitudes on sides \( b \) and \( c \) respectively. **Suggestion:** Look for a pair of similar triangles.

*11. Prove the proposition in Ex. 10 when one of the angles of the triangle is a right angle. Prove this same proposition when one of the angles is obtuse. Draw a figure in each case.

*12. The preceding proposition can be generalized to include all three altitudes, as follows: \( ag = bh = eh \). That is, the product of an altitude and the corresponding side of a given triangle is constant for the triangle. Prove it.

13. Draw a triangle on any convenient sphere (a tennis ball will do). * Then on the same sphere draw a second triangle having one side three times as long as a side of the first triangle. Make the angle at each end of this side in the second triangle equal to the corresponding angle in the first triangle. Will these two triangles be similar?

14. Does Principle 6 (Case 2 of Similarity) apply to triangles on a spherical surface?

**PRINCIPLE 7**

The truth of the principle stated below seems so obvious that you may omit the analysis and proof for the present. You will be asked to consider them later.

**Principle 7.** If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely, if two angles of a triangle are equal, the sides opposite these angles are equal.

—See note for Ex. 5, page 61.
GIVEN: Triangle $ABC$ (Fig. 7) in which $AC=BC$.

TO PROVE: $\angle CBA = \angle CAB$.

ANALYSIS: The Principle of Angle Measure (Principle 3) will not apply here because the half-lines that form the angles $CBA$ and $CAB$ are not numbered; and Cases 1 and 2 of Similarity have to do with two triangles. In this special case, however, we may regard the triangle $ABC$ in two different ways because of the fact that $AC=BC$. In general, a broken line $PQR$ (Fig. 8) is not the same as the broken line $RQP$: for the first distance, $PQ$, of $PQR$ will not ordinarily be equal to the first distance, $RQ$, of $RQP$; nor will the second distances ordinarily be equal, either. If $PQ$ equals $RQ$, however, we can apply Case 1 of Similarity; the factor of proportionality, $k$, is 1.

PROOF: The given triangle may be regarded in two ways: as the triangle $ACB$ and as the triangle $BCA$ (Fig. 9).

In triangles $ACB$ and $BCA$ respectively, $AC=BC$ (Given), $\angle ACB = \angle BCA$ (For each represents the same angle between the same two line segments), and $CB=CA$ (Given).

Therefore $\angle CBA = \angle CAB$ (Case 1 of Similarity).

To prove the converse, we proceed as follows.

GIVEN: Triangle $ABC$ (Fig. 7) in which $L CAB = L CBA$.

TO PROVE: $BC=AC$. 
ANALYSIS: We may regard triangle $ABC$ in two ways again, this time making use of Case 2 of Similarity.

PROOF: In the triangles $CAB$ and $CBA$ respectively (Fig. 10),

- $\angle CAB = \angle CBA$ (Given),
- $AB = BA$,
- and $\triangle ABC \cong \triangle BAC$ (Given).

Therefore $BC = AC$ (Case 2 of Similarity).

This theorem affords us a fourth method of proving two distances equal or two angles equal.

A triangle with two sides equal is called an **isosceles** triangle. The word “isosceles” means “equal legs.”

**EXERCISES**

*1. Prove that if all three sides of a triangle are equal, the three angles of the triangle are equal also.

**GIVEN:** Triangle $ABC$ in which $AB = BC = CA$.

**TO PROVE:** $\angle A = \angle B = \angle C$.

**PROOF:** Now prove that the theorem is true. You should be able to do this without help.

*2. Prove that if all three angles of a triangle are equal, the sides of the triangle are equal also.

A triangle in which all three sides are equal is called an **equilateral** triangle. Since its three angles are also equal, such a triangle is also said to be **equiangular**.

*3. In Fig. 11 on page 78 two isosceles triangles, $ABD$ and $CBD$, have a side $BD$ in common. Prove that $\angle ABC = \angle ABD$. 

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4. In Fig. 11 prove that a line drawn through A and C bisects $L BCD$.

5. Two isosceles triangles, $BKD$ and $BCD$ (Fig. 12), have a side $BD$ in common. Prove that $LKBC=LKDC$.

6. In Fig. 13 $Lp=Lq$. Prove that $AB=BC$.

7. In Fig. 14, $AB=BD$ and $AC=CD$. Prove that $LBAC=LBDC$, that $LABC=LBDC$, and that $AD$ is perpendicular to $BC$.

8. Is Principle 7 true of triangles on a spherical surface?

**PRINCIPLE 8**

Principle 8 (Case 3 of Similarity) is presented on page 79. Omit the proof of Principle 8 for the present, since you will be asked to consider it later. Compare Principles 5, 6, and 8 to get clearly in mind how Cases 1, 2, and 3 of Similarity differ. These three cases supply three important methods of proving triangles similar.
Principle 8. Case 3 of Similarity. Two triangles are similar if their sides are respectively proportional.

Given: Triangles $ABC$ and $A'B'C'$ (Fig. 15) in which $A'B'=k\cdot AB$, $B'C'=k\cdot BC$, and $C'A'=k\cdot CA$.

To prove: Triangle $A'B'C'$ similar to triangle $ABC$.

Fig. 15

Analysis: All we need to do in order to prove this proposition is to show one pair of corresponding angles equal, for we can then make use immediately of Case 1 of Similarity. But none of the principles already established would appear to be of any help in this respect, unless possibly we can manage to apply the one that we have just proved. Let us try that.

Proof: On the side of $A'B'$ opposite from $C'$ construct the angle $A'B'C''$ equal to angle $ABC$, and lay off $B'C''$ equal to $k\cdot BC$. Draw $C'A'$.

By applying Case 1 of Similarity to triangle $A'B'C''$ and triangle $ABC$, we see that $\angle A'C''B' = \angle LACB$, and $A'C'' = k\cdot AC$. Show this in detail.
We shall show that angle \(A'C''B'\) = angle \(A'C'B'\), and hence that angle \(ACB\) = angle \(A'C'B'\). Draw \(C'C''\).

In triangle \(A'C'C''\) we know that
\[A'C'' = k \cdot AC \text{ (Just proved)},\]
and \(A'C' = k \cdot AC \text{ (Given)}\).

Therefore \(A'C' = A'C''\), and \(LA'C'C'' = LA'C'C'\) (By Principle 7).

In triangle \(B'C'C''\)
\[B'C' = k \cdot BC \text{ (Given)},\]
and \(B'C'' = k \cdot BC \text{ (By construction)}\).

Therefore \(B'C' = B'C''\), and \(LB'C'C'' = LB'C'C'\) (By Principle 7).

Therefore \(LA'C'B'\), which is the sum of (difference between) \(LA'C'C''\) and \(LB'C'C''\), is equal to \(LA'C'B'\), which is the sum of (difference between) \(LA'C'C'\) and \(LB'C'C'\).

But we have already proved that \(LA'C'B' = LACE\). Therefore \(LA'C'B' = LACB\) also.

By applying now Case 1 of Similarity to triangles \(ABC\) and \(A'B'C'\), it follows that these triangles are similar.

In other words, if in triangles \(ABC\) and \(A'B'C'\), \(A'B' = k \cdot AB\), \(B'C' = k \cdot BC\), and \(C'A' = k \cdot CA\), then \(LA' = LA\), \(LB' = LB\), and \(LC' = LC\).

Discuss the case when \(B'\) happens to lie on \(C'C''\).

In general, Principles 5, 6, and 8 have to do with similar triangles; in the special case \(k = 1\), however, these principles have to do with triangles which are not only similar but also equal in all respects.

**EXERCISES**

1. If every line in Fig. 16 on page 81 is made three times as long, which angles will remain unchanged? Which angles can vary in size? Why?
2. If three sticks are nailed together to form a triangle with only one nail at each vertex, as shown in Fig. 17, will such a triangle be rigid? Try it and see. Why should this be so?

3. Mr. Lee’s front gate was sagging; so he braced it with a diagonal strut, as shown in Fig. 18. Why did this help? What mathematical principle was used?

4. Five sticks are nailed together to form a pentagon; there is only one nail at each vertex. How many cross-braces from corner to corner do you need to make the figure rigid? In how many ways can you do this?

*5. Prove that the line drawn from the vertex of an isosceles triangle to the mid-point of the opposite side bisects the angle at the vertex and is perpendicular to the opposite side. That is, given triangle $ABC$ (Fig. 19), in which $AC=BC$ and $AM=MB$, prove $\angle ACM=\angle BCM$. Also prove $\angle BMC=90^\circ$.

*6. Consider Ex. 5 in connection with Ex. 16, page 63, and then prove the following proposition: Every point in the perpendicular bisector of the line segment $AB$ is equidistant from $A$ and $B$; and conversely, every point that is equidistant from $A$ and $B$ lies in the perpendicular bisector of $AB$. 
7. Draw a triangle with three equal sides—all small—on a large globe (or a tennis ball). Measure its angles as best you can with a protractor. Now enlarge the triangle, keeping the sides all equal, so that it reaches from equator to pole. What is the measure of the angles of the second triangle?

8. Does Principle 8 apply to triangles on a spherical surface?

PRINCIPLE--

The truth of the next principle is not so obvious as that of Principles 6, 7, and 8. Principle 9 is stated for you below, and then part of the proof is discussed. You are asked to complete the proof.

**Principle 9.** The sum of the three angles of a triangle is 180°.

**GIVEN:** Triangle $ABC$ (Fig. 21).

**TO PROVE:** $LA + LB + LC = 180°$.

*See note for Ex. 5, page 61.*
ANALYSIS: In Ex. 17, page 63, we were unable to prove one of the four small triangles (namely, triangle $KLM$ in Fig. 21) similar to triangle $ABC$ because we had not at that time proved Case 3 of Similarity. With Case 3 to help us we can now show that $\angle KLM = \angle C$, and so prove this very important principle concerning the sum of the angles of a triangle.

PROOF: Let $K$, $L$, and $M$ be the mid-points of $BC$, $CA$, and $AB$ respectively. 

In triangles $MBK$ and $ABC$,

$MB = \frac{1}{2} AB$ (By construction),
$\angle MBK = \angle ABC$,
and $BK = \frac{1}{2} BC$ (By construction).

Therefore $MK = \frac{1}{2} AC$ and $\angle BMK = \angle A$ (By Case 1 of Similarity).

By using triangles $AML$ and $ABC$, you can prove in similar manner that $ML = \frac{1}{2} BC$ and $\angle LMA = \angle B$. Prove this. Also prove that $LK = \frac{1}{2} AB$.

Since the sides of triangle $KLM$ have been shown to be equal respectively to one-half the sides of triangle $ABC$, it is clear (from Case 3 of Similarity) that $\angle KLM = \angle C$.

We have shown that $\angle A = \angle BMK$, $\angle C = \angle KLM$, and $\angle B = \angle LMA$.

Therefore $\angle A + \angle C + \angle B = \text{straight angle } BMA = 180^\circ$.

We can illustrate the fact that the sum of the angles of a triangle is $180^\circ$ by moving a pencil around any triangle in the way shown in Fig. 22. The pencil is first placed in position 1. Then it is rotated counter-clockwise through angle $A$ to position 2. Next the pencil is moved along line $AB$ to position 3. It is

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*We know by Principle 1 that these mid-points exist and can be found.
rotated through what angle to reach position 4? Along what line is it then moved to reach position 5? Through what angle is it rotated to reach position 6? Is the pencil now in the same line as at the start? Through how many degrees has it turned?

We could also prove the converse of Principle 9, namely, that if we have three angles whose sum is $180^\circ$ there is a triangle having these three angles. The proof is not difficult; but it is quite long, and we shall not stop for it.

A triangle all of whose angles are acute is called an acute triangle.

A triangle one of whose angles is a right angle is called a right triangle.

A triangle one of whose angles is obtuse is called an obtuse triangle.

EXERCISES

1. Prove that triangle $KLM$ in Fig. 21, page 82, is equal to each of the triangles $AML$, $MBK$, and $LKC$.

2. Prove that in triangle $ABC$ (Fig. 23) the exterior angle $BCD$ is equal to the sum of the two remote interior angles, $0 A$ and $B$.

3. Prove that the sum of the angles of a quadrilateral is $360^\circ$. **Suggestion:** Divide the quadrilateral into two triangles.

4. Prove that the sum of the angles of a pentagon is $3$ times $180^\circ$, or $540^\circ$.

5. Prove that the sum of the angles of a hexagon is $4$ times $180^\circ$, or $720^\circ$. 

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6. Prove that the sum of the angles of a convex polygon of \( n \) sides is \((n-2)\ 180^\circ\).

A **polygon in which all sides are equal** is called an **equilateral polygon**.

A **polygon in which all angles are equal** is called an **equiangular polygon**.

Any **polygon that is both equilateral and equiangular** is called a **regular polygon**.

7. How many degrees are there in each of the angles of a regular pentagon?

8. How many degrees are there in each of the angles of a regular hexagon?

9. How many degrees are there in each of the angles of a regular octagon?

*10. How many degrees are there in each of the angles of a regular polygon of \( n \) sides?

11. Each angle of a regular polygon is \( 144^\circ\). How many sides has it?

*12. Prove that two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

In a right triangle the side opposite the right angle is called the **hypotenuse**. We also use the word "hypotenuse" to mean the length of this side, but we shall do this only when there is no likelihood of confusion.

*13. Prove that two right triangles are equal if the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.

A number \( m \) is the mean proportional between two other numbers \( a, b \) when \( \frac{a}{m} = \frac{m}{b} \), that is, when \( m^2 = ab \).
14. Prove that the altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse. That is, in Fig. 24 prove that \( h^2 = mn \).

15. In the right triangle \( ABC \) in Fig. 25, \( CD \) is drawn perpendicular to \( AB \). Prove that \( b^2 = cm \) and that \( a^2 = cn \).

16. The altitude on the hypotenuse of a right triangle divides the hypotenuse into segments that are 5 inches and 8 inches long. Find the length of the altitude on the hypotenuse and the length of the other two sides of the triangle.

17. The sides including the right angle of a right triangle are 5 inches and 12 inches long. Find the length of the altitude and of the segments into which it divides the hypotenuse. *Suggestion:* Of the basic principles that have been considered so far in this book, only those concerning similar triangles can be of any help here. In fact, almost every exercise in this book is based either immediately or ultimately on similar triangles.

18. The altitude on the hypotenuse of a right triangle divides the hypotenuse into segments that have the ratio 1 to 3. Find the ratio of the other two sides of the triangle.

19. If in Ex. 18 the segments have the ratio 1 to \( k \), find the ratio of the other two sides.

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20. If in Ex. 18 on page 86 the segments have the ratio $p : q$, find the ratio of the other two sides.

21. The sides including the right angle of a right triangle have the lengths $a$ and $b$. Find the length of the altitude and the ratio of the segments into which it divides the hypotenuse.

An exterior angle of a convex polygon is the angle between the prolongation of one side of the polygon and the next succeeding side.

*22. Measure carefully the exterior angles of the irregular 5-sided polygon in Fig. 26. Then draw an irregular 6-sided polygon and measure its exterior angles carefully. Do the same for an irregular 7-sided polygon. Are you ready now to state a theorem concerning the sum of the exterior angles of any convex polygon? See if you can prove your theorem, being careful to note that the sum of corresponding interior and exterior angles at a vertex is always equal to 180°.

23. A man owns an irregular field of eight sides, $ABCDEFGH$, as shown in Fig. 27. By means of a transit he measures the angles at $B$, $C$, $D$, ..., $H$, as indicated in the diagram. He checks his work by measuring the exterior angle at $A$. What should this angle at $A$ equal?
**PRINCIPLE 10**

Although the truth of the next principle seems obvious, do not take it for granted. Study the proof carefully.

*Principle 10.* All points equidistant from the endpoints of a line segment, and no others, lie on the perpendicular bisector of the line segment.

**GIVEN:** Line segment \( AB \) and any point \( P \) such that \( AP = BP \). See Fig. 28.

**TO PROVE:** (1) \( P \) lies on the perpendicular bisector of \( AB \).

(2) Any point \( Q \) on the perpendicular bisector of \( AB \) will be equidistant from \( A \) and \( B \).

Notice that the statement "All points equidistant . . . , and no others, lie on the perpendicular bisector" also means "If and only if a point is equidistant . . . , it lies on the perpendicular bisector." We have seen (page 28) that such a statement is equivalent to a proposition and its converse.

**ANALYSIS:** To prove that \( P \) lies on the perpendicular bisector of \( AB \), we may not draw a line from \( P \) perpendicular to \( AB \) and show that the mid-point of \( AB \) lies on this perpendicular; for we need Principle 11 before we can do this. We may, however, connect \( P \) and the mid-point \( M \) of \( AB \) and show that \( PM \) is perpendicular to \( AB \).

**PROOF:** (1) Draw \( PM \) from \( P \) to the mid-point \( M \) of \( AB \). In triangles \( AMP \) and \( BMP \), \( AP = BP \), \( AM = BM \), and \( MP = MP \); so \( LAMP = LBMP \). Why? Each of these angles is a right angle. Why? So \( PM \) is the perpendicular bisector of \( AB \).
(2) We must now show that this perpendicular bisector contains no points that are not equidistant from $A$ and $B$. Let $Q$ be any point on the perpendicular bisector $PM$, however far extended. Then $QA = QB$. Why?

**PRINCIPLE 11**

The truth of the next principle seems obvious; so omit the proof now. You will be asked to consider it later.

*Principle 11.* Through a point not on a line there is one and only one perpendicular to the line.

**GIVEN:** Line $l$ and point $P$ not on $l$. See Fig. 29.

**TO PROVE:** (1) There can be only one perpendicular to $l$ through $P$.

(2) There can be only one such perpendicular.

**ANALYSIS:** We know from the Principle of Angle Measure (Principle 3) that at every point of $l$ there can be one and only one perpendicular to $l$; but we have no assurance that a given point $P$ not on $l$ lies on anyone of these perpendiculars. In order to prove this we must rely on one or more of the preceding principles or on theorems that we have met in the exercises. Of all of these, Principle 10 seems most likely to be helpful. Let us try that first.

**PROOF:** (1) Let $A$ and $B$ be any two distinct points on $l$. Draw $PA$. Then construct angle $P'AB$ equal to angle $PAB$ in such a manner that $P$ and $P'$ lie on opposite sides of $l$. Make $PIA = PA$. Draw $PB$ and $P'B$.

From Principle 5 (Case 1 of Similarity) we see that triangles $PAB$ and $P'AB$ are similar and that $PB = P'B$. 89
Since $A$ is equidistant from $P$ and $P'$, and $B$ is also, line $l$ must be the perpendicular bisector of $PP'$. There is, therefore, at least one perpendicular to $l$ through $P$.

(2) Now let us assume that there are two lines through $P$ both perpendicular to the line $l$, one meeting $l$ at the point $D$ and the other meeting $l$ at the point $E$. See Fig. 30. Since two angles of the triangle $PDE$ must be right angles, the third angle, $LDPE$, must be $0^\circ$ in order that the sum of all three angles shall be $180^\circ$. Thus the lines $PD$ and $PE$ must coincide and there cannot be two distinct perpendiculars from $P$ to $l$.

**PRINCIPLE 12**

The next principle is an interesting and very important one that was known to mathematicians as long ago as 2600 B.C. Study the proof of this principle carefully.

*Principle 12. The Pythagorean Theorem.* In any right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides; and conversely.

**GIVEN:** Triangle $ABC$ (Fig. 31) in which $\angle C = 90^\circ$.

**TO PROVE:** $(AB)^2 = (AC)^2 + (BC)^2$, or $c^2 = a^2 + b^2$.

**ANALYSIS:** We cannot analyze this proposition by assuming that the conclusion is true and showing therefrom that $\angle C$ must be a right angle. The squared numbers suggest that we might try to find a series of mean proportions. The
proof of this proposition was probably not arrived at in that way, however. More likely the proof was established in the same way that the proposition itself was discovered, namely, by successive generalizations based on observation and contemplation of many special cases.

There are many methods of proving this theorem. The following special example shows the method we shall follow.

In Fig. 32 notice that we can enlarge the right triangle $A$ to form triangle $B$ and also triangle $C$. The sides of triangle $B$ are how many times as long as the corresponding sides of triangle $A$? How do the sides of triangle $C$ compare with those of triangle $A$? If we place triangles $B$ and $C$ together so that their equal sides coincide, we can form a right triangle that is similar to each of these two triangles. The sides including the right angle in this composite triangle are $x$ times the corresponding sides of triangle $A$. But we could have enlarged triangle $A$ directly to form a triangle with each side $x$ times as large as the corresponding side of triangle $A$, as shown by triangle $D$ in Fig. 32. Since triangle $D$ is just like the composite triangle above it, we see that $x^2 = 9 + 16$ and $x = 5$. Thus the hypotenuse of triangle $A$ is 5, and $5^2 = 3^2 + 4^2$.
PROOF: Let the lengths of the sides of the given triangle $ABC$ be $a, b, c$, as shown in Fig. 33.

This triangle may be enlarged so that each side of the new triangle $A'B'C'$ is $b$ times as large as formerly. See triangle $A'B'C'$ in Fig. 33.

Now construct triangle $B'C'D'$ so that $L C'B'D'= LA$ and $LB'C'D'=90^\circ$. See Fig. 33. It follows from Case 2 of Similarity that each side of triangle $B'C'D'$ is $a$ times the corresponding side of triangle $ABC$.

![Fig. 33]

Now prove that $LA'B'D'=90^\circ$. Since $LA'B'D'=LC$ (Why?), and since $LA'=LA$ by construction, two angles of triangle $A'B'D'$ are equal respectively to two angles of triangle $ABC$, and therefore triangles $A'B'D'$ and $ABC$ are similar, by Case 2 of Similarity.

But $A'B'=cb=c\cdot AC$, indicating that the factor of proportionality is $c$.

Therefore $A'D'=c\cdot AB$, or $c^2$.

But $A'D'$ is also equal to $b^2+a^2$.

Therefore $c^2 = b^2+a^2$.

To prove the converse of this proposition, we proceed as follows.

**GIVEN:** Lengths $a, b, c$ such that $c^2 = b^2+a^2$.

**TO PROVE:** There is one and only one right triangle having these lengths for sides.
PROOF: Construct a triangle $ABC$ such that $AC=b$, $LC=90°$, and $CB=a$. Then $(AB)^2 = b^2 + a^2$ (Why?)

But $c^2 = b^2 + a^2$ (Given).

Therefore $(AB)^2 = c^2$ and $AB = c$.

So there exists at least one right triangle with sides $a$, $b$, $c$. If more than one such triangle exists, these triangles must be equal, for if they are unequal they violate Principle 8.

Several propositions follow so immediately from the Pythagorean Theorem that almost no proof is required to establish them. Any such proposition is called a COROLLARY of the parent theorem.

**Corollary 12a.** If the hypotenuse and another side of two right triangles are in proportion, the two triangles are similar.

**GIVEN:** Triangles $ABC$ and $A'B'C'$ (Fig. 34) in which $LC'=LC=90°$, $c'=k·c$, and $a'=k·a$.

**TO PROVE:** Triangles $ABC$ and $A'B'C'$ similar.

**PROOF:** If we simply show that $b' = k·b$, it follows from Case 3 of Similarity that the triangles are similar.

$b'^2 = C'2 - a'^2$ (By the Pythagorean Theorem).

But $C'^2 - a'^2 = k^2·c^2 - k^2·a^2 = k^2(c^2 - a^2)$.

and $c' = a'$. $b'^2 = b^2$ (By the Pythagorean Theorem).

Therefore $b' = k·b$ and $b' = k·b$.

**Corollary 12b.** Two right triangles are equal if the hypotenuse and another side of one are equal respectively to the hypotenuse and another side of the other.
Corollary 12c. The sum of two sides of a triangle is greater than the third side.

GIVEN: Triangle ABC (Fig. 35).

TO PROVE: \( AB + BC > AC \).

PROOF: Through B draw BD perpendicular to AC and meeting AC, or AC extended, at D.
\[
(AB)^2 = (AD)^2 + (BD)^2 \quad \text{(By the Pythagorean Theorem)}.
\]
Therefore \((AB)^2 > (AD)^2\), and \( AB > AD \).
Similarly \( BC > DC \).
Therefore \( AB + BC > AD + DC \), and \( AB + BC > AC \).

Show that this proof holds when \( D \) falls on A or C; also when \( A \) is between \( D \) and \( C \), and when \( C \) is between \( A \) and \( D \).

*To THE TEACHER: As stated in the Preface, this geometry takes for granted the Laws of Number, meaning thereby the fundamental relations between real numbers and the operations involving real numbers. These all seem reasonable to the student from his previous experience in arithmetic and algebra, though he has probably never formulated them definitely and perhaps would hardly recognize these reasonable ideas of his if he should see them stated as mathematicians state them. There is little to be gained by asking the student to adopt these technical expressions at this time. It is better for him merely to agree, as here, that common sense demands that \( AB + BC \) shall be greater than \( AD + DC \). It is to be hoped, however, that the student will become increasingly curious about the foundations of the system of real numbers, of which he makes daily use. For convenience, therefore, these fundamental ideas are listed at the back of the book. The particular "law" that bears on the addition of inequalities is the next to the last in this list.
Corollary 12d. The shortest distance from a point to a line is measured along the perpendicular from the point to the line.

Given: Point P, line AB, and PD perpendicular to AB. See Fig. 36.

To prove: PD is less than any other line through P to AB, such as PE.

Proof: \((PE)^2=(PD)^2+(ED)^2\) (By the Pythagorean Theorem).

Therefore \((PE)^2>(PD)^2\), and \(PE>PD\).

Hereafter when we speak of the distance from a point to a line, we shall mean the perpendicular distance, for example, \(PD\) in Fig. 36. The point D in Fig. 36 is called the foot of the perpendicular from \(P\) to \(AB\).

Corollary 12e. Of two oblique lines drawn from a point to a line, the more remote is the greater; and conversely.

Given: Point P, line AB, PD perpendicular to AB, and \(FD>ED\). See Fig. 37.

To prove: \(PF>PE\).

Supply the complete proof. Also supply the complete proof of the converse.

Exercises

1. Find the hypotenuse of a right triangle, given the other sides as in (a) to (p) on the next page. Limit your answers to significant figures.*

*This means that the answer should be stated to the same degree of accuracy as the numbers that are given.
2. Given the hypotenuse and another side of a right triangle as in (a) to (k) below, find the third side. Limit your answers to significant figures.

(a) 17, 8  (d) \( s, \sqrt{2}, 1 \)  (g) \( s, \frac{s}{2} \)  (j) 9.4, 3.5
(b) 2, 1  (e) \( 2s, s \)  (h) \( s\sqrt{2}, s \)  (k) 204, 111
(c) 2, \( \sqrt{3} \)  (t) \( 2s, s\sqrt{3} \)  (i) 18, 13

3. The road from A to B goes due east for 17 miles and then goes due north for 11 miles more. How far is it from A to B in a straight line?

4. By means of the diagrams in Fig. 38 prove in two ways that in a 30°-60° right triangle the shortest side is equal in length to half the hypotenuse.

5. Prove that a right triangle in which the shortest side is half the hypotenuse is a 30°-60° right triangle. Try to do this in two ways.

6. Prove that the lengths of the sides of a 30°-60° right triangle are in the ratio 1: \( \sqrt{3} : 2 \).
7. Prove that a triangle whose sides are in the ratio $1 : \sqrt{3} : 2$ is a $30^\circ$-$60^\circ$ right triangle.

8. In triangle $ABC$, $AB = 5$, $BC = 12$, and $CA = 13$. How many degrees are there in $\angle ABC$?

9. In triangle $DEF$, $DE = 6 = EF$, and $FD = 6\sqrt{2}$. How many degrees are there in each of the three angles of the triangle?

10. In triangle $GRK$, $GH = 4$, $HK = 4\sqrt{3}$, and $KG = 8$. How many degrees are there in each of the three angles of the triangle?

11. A certain box is 12 inches long, 4 inches wide, and 3 inches high. How long is the diagonal on each of the six faces of the box?

12. Find the length of the diagonal that passes through the center of the box in Ex. 11.

13. Show that a triangle with sides $p^2 - q^2$, $2pq$, and $p^2 + q^2$, where $p$ and $q$ are any positive whole numbers you like, provided $p$ is greater than $q$, is a right triangle.

14. Find the numbers $p$ and $q$ that lead to the $3, 4, 5$ right triangle.

15. Find the numbers $p$ and $q$ that lead to the $5, 12, 13$ right triangle.

16. Make a table showing ten triplets of numbers that are the sides of right triangles.

17. Using the method described in the analysis of the Pythagorean Theorem (Principle 12), show that the right triangle having its two shorter sides equal to 5 and 12 has a hypotenuse of 13.

18. Show how you would derive Corollary 12c from Corollary 12d, in case the latter had been stated first.

19. Prove the Pythagorean Theorem by means of the relations established in Ex. 15, page 86.
Trace this alternate proof of the Pythagorean Theorem back through Principles 11 and 9 and other principles to Principle 5. Show similarly that the proof of the Pythagorean Theorem given on page 92 also depends ultimately upon Principle 5.

20. Prove that the difference between two sides of a triangle is less than the third side. *Suggestion:* Use Corollary 12c.

21. In Fig. 37, page 95, prove that \( LDEP > LDFP \).

22. Without using the Pythagorean Theorem prove that the hypotenuse of an isosceles right triangle will have the length \( \sqrt{2} \) if each of the equal legs has the length 1. *Suggestion:* Consider the similar triangles in Fig. 39.

23. The ancient Greeks regarded the Pythagorean Theorem as involving areas, and they proved it by means of areas. We cannot do so now because we have not yet considered the idea of area. Assuming for the moment, however, the idea of the area of a square, use this idea instead of similar triangles and proportion in Ex. 22 above to show that \( x = \sqrt{2} \).

**SUMMARY**

6. **Case 2 of Similarity.** Two triangles are similar if two angles of one are equal to two angles of the other.

**DEFINITION:** *altitude of a triangle*

7. If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely, if two angles of a triangle are equal, the sides opposite these angles are equal.

**DEFINITIONS:** isosceles, equilateral, equiangular

8. **Case 3 of Similarity.** Two triangles are similar if their sides are respectively proportional.
9. The sum of the three angles of a triangle is 180°.

**Definitions:** acute triangle, right triangle, obtuse triangle, regular polygon, hypotenuse, mean proportional, exterior angle of a convex polygon

10. All points equidistant from the end-points of a line segment, and no others, lie on the perpendicular bisector of the line segment.

11. Through a point not on a line there is one and only one perpendicular to the line.

12. *The Pythagorean Theorem.* In any right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides; and conversely.

**Definition:** corollary

12a. If the hypotenuse and another side of two right triangles are in proportion, the two triangles are similar.

12b. Two right triangles are equal if the hypotenuse and another side of one are equal respectively to the hypotenuse and another side of the other.

12c. The sum of two sides of a triangle is greater than the third side.

12d. The shortest distance from a point to a line is measured along the perpendicular from the point to the line.

**Definition:** distance from a point to a line

12e. Of two oblique lines drawn from a point to a line, the more remote is the greater; and conversely.

When we wish to prove two distances equal, we shall rely on Principles 1, 5, 6, 7, 10, 12, and 12b, and on Ex. 13, page 85.

When we wish to prove two angles equal, we shall rely on Principles 3, 5, 6, 7, 8, 9, and 12b.
EXERCISES

When an "original" proposition baffles you, draw an exact figure; then state as carefully as you can exactly what you can prove and what you cannot prove. This is the method of the scientist.

1. The perimeters of two similar polygons are 19 inches and 33 inches. If a side of the first polygon is 4 inches, how long is the corresponding side of the second?

2. The sides of a polygon are 3.1 cm., 2.7 cm., 4.6 cm., 4.4 cm., 5.0 cm., and 2.2 cm. Find the perimeter of a similar polygon whose longest side is 7.0 cm.

3. The perimeter of an isosceles triangle is 21, and one of the equal sides is \( \frac{5}{6} \) the third side. Find the three sides.

4. Enlarge the spiral in Fig. 40 in the ratio 3 to 2, noting that in general the distance \( OP \) is proportional to \( LAOP \). In Fig. 40 \( LAOB=30^\circ \), \( OB=\frac{1}{3} \) in.; \( LAOC=60^\circ \), \( OC=\frac{2}{3} \) in.; \( LAOD=90^\circ \), \( OD=\frac{3}{8} \) in.; \( LAOE=120^\circ \), \( OE=\frac{1}{8} \) in., etc.

5. Prove that if alternate vertices of a regular hexagon be joined by straight lines, the triangle formed will have each angle equal to 60\(^\circ\).
6. The sides of a regular pentagon are extended to form a five-pointed star. Find all the angles of the star.

7. Find all the angles of a regular six-pointed star.

8. The sides of a regular polygon of \( n \) sides are extended to form a star. Show that the angle at each point of the star is \( \frac{n}{n-5} \cdot 180^\circ \).

9. From the mid-point of the hypotenuse of a right triangle perpendiculars are drawn to the other two sides of the triangle. Prove that these lines are perpendicular bisectors of the sides.

10. Prove that in any right triangle the mid-point of the hypotenuse is equidistant from all three vertices.

11. Prove that in any right triangle the point of intersection of the perpendicular bisectors of the two shorter sides is the mid-point of the hypotenuse.

12. Find the length of the diagonals of a cube one edge of which is \( s \).

13. Prove that diagonals \( AG \) and \( CE \) of the cube in Fig. 41 form angles that are not equal to any angle formed by diagonals \( AC \) and \( BD \) of the square \( ABCD \).

14. Prove that any pair of diagonals of a cube meet at the same angle. For example, show that the angle formed by \( AG \) and \( EC \) in Fig. 41 is equal to the angle formed by \( EC \) and \( HB \).

15. Find the angle between the diagonals of a cube. (Use trigonometric tables.)
16. Prove that of all the straight lines that can be drawn from the vertex $G$ of the cube in Fig. 42 to points in the opposite face $ABCD$, the line that makes the smallest angle with the face $ABCD$ is the diagonal $GA$. See Ex. 21, page 98.

17. In the house shown in Fig. 43, how far is it from $A$ to $B$?

18. Find angle $APQ$ in the house shown in Fig. 43. This angle is called the angle of pitch of the rafters.

19. Each slanting edge of a pyramid is 6 inches long; the base is a square 4 inches on a side. Find the altitude of the pyramid.

20. A tomato can is 5 inches high and 4 inches in diameter. How far is it from the center of the can to a point on the rim?

21. How far from $A$ is the center of the endless broken-line spiral of Fig. 44 on the next page? Prove that you are right. Suggestion: Calculate the directed distances $AC, CE, EG, GI \ldots$ and note that the sum of the infinite series of steps, $AC + CE + EG + GI + \ldots$, yields the distance from $A$ to the point in question. You can easily learn how to find the sum of an infinite series of this
sort with alternating signs by turning to an algebra text, 

For example, \( \frac{1}{5} - \frac{1}{25} + \frac{1}{125} - \cdots \)\( \frac{1}{5} \frac{1}{25} \frac{1}{125} \cdots \)
This diagonal scale gives us lengths in inches, tenths, and hundredths. For example, PO = 2.76 inches. The construction of the scale is based on this theorem: Two or more transversals are cut into proportional segments by a system of parallels.

Notice the plan followed in naming the streets and the avenues of Parkton. The streets and avenues of this city form a rectangular network.
CHAPTER 4

Pal'allel Lines and Netwol'ks

The majority of geometric relations cluster around the ideas "equal" and "similar." That is why the simplest notions about equal and similar triangles are taken as the basis of this geometry. From these simplest notions we derive the more complicated relations of geometry.

Principles 10, 11, and 12 in Chapter 3 state important relationships about perpendiculars and about right triangles. In this chapter we shall learn more about perpendiculars. We shall also learn about parallels, which are closely related to perpendiculars.

Parallel Lines

Long before you began this course in demonstrative geometry you had learned to think of "parallel lines" as lines which lie in the same plane and do not meet, however far extended. Probably it never occurred to you to inquire, first of all, whether such lines exist or not. In this development of geometry, however, we prefer to make sure that such lines exist before we define them or discuss any of their relationships. Consequently our
first theorem in this chapter, Theorem 13, states that through a given point not on a given line there is one and only one line which does not meet the given line. Then, having proved this theorem, we describe such lines, by definition, as parallel. In general we prefer not to define anything until we have first shown that it exists. However, to follow this practice without exception throughout our study of geometry would make us consider many difficult and annoying details. It is desirable, nevertheless, to mention this ideal and to illustrate it with this theorem.

In other geometry textbooks the proposition stated in Theorem 13 is taken as an assumption. It is often called the Parallel Postulate. By means of it these other books prove the propositions concerning similar triangles and the sum of the angles of a triangle. We have assumed instead a proposition concerning similar triangles. We have called this proposition Case 1 of Similarity (Principle 5). By using this principle we can prove, or deduce, this proposition concerning parallels. As in the case of Principles 6, 7, 8, and 11, the proof of Theorem 13 may be omitted at this time; we shall return to it in Chapter 10. A glance at the proof reveals that it depends in part upon Principle 11, which was proved by means of Principles 5 and 9. Principle 9, in turn, was proved by means of Principles 5 and 8; while Principle 8 depends upon Principle 7, and Principle 7 upon Principle 5. So Theorem 13 is based fundamentally upon Principle 5. Our geometry needs to assume only five fundamental propositions because these five are very powerful, as we have just observed in the use of Principle 5.

These theorems on parallels and perpendiculars are stated for the two-dimensional cases only. The exercises afford opportunity for the extension of these principles to three dimensions.
Theorem 13. * Through a given point not on a given line there is one and only one line which does not meet the given line.

---

**GIVEN:** Line \( l \) and point \( P \) not on \( l \). See Fig. 1.

**TO PROVE:** (1) There is at least one line through \( P \) which does not meet line \( l \).

(2) There can be not more than one such line through \( P \).

**ANALYSIS:** The words "one and only one" are familiar to us from our experience with perpendiculars; so it is natural to depend on perpendiculars in our efforts to prove the first part of this theorem.

**PROOF:** (1) There is one and only one line through \( P \) which is perpendicular to line \( l \) (by Principle 11); call it \( PD \). And there is one and only one line through \( P \) which is perpendicular to \( PD \) (by Principles 3 and 4); call it \( m \). Lines \( l \) and \( m \) are both perpendicular to \( PD \) and so cannot meet; for if they had a point in common, we should have two lines through that point both perpendicular to \( PD \), and that is impossible (by Principle 11).

Therefore, line \( m \) does not meet line \( l \), and there is at least one line through \( P \) which does not meet \( l \).

---

*The five fundamental assumptions and the seven basic theorems in Chapters 2 and 3 are called principles because they are the backbone of our geometry. For convenience these twelve principles and the subsequent theorems are numbered consecutively.*
It is necessary now to prove that any other line through \( P \) must meet \( l \).

Let \( n \) be a line through \( P \) which makes with \( PD \) an angle less than 90°. From a point \( Q \) on \( n \) (other than \( P \)) draw a perpendicular to \( PD \) meeting it at \( R \). In Fig. 1, \( R \) is shown between \( P \) and \( D \), but this is not necessary; \( R \) can be any distance from \( P \) and on either side of \( P \).

Measure the distances \( PR, RQ, \) and \( PD \). From these measurements compute the distance \( DB \) such that \( \frac{PR}{PD} = \frac{RQ}{DS} \). Then from \( D \) measure off the distance \( DS \) on line \( l \) and draw \( PS \).

By Case 1 of Similarity (Principle 5) we know that triangle \( PDS \) is similar to triangle \( PRQ \). Consequently \( \angle DPS = \angle RPQ \), and lines \( PS \) and \( n \) coincide. Therefore \( n \) must meet \( l \), and there is only one line through \( P \) which does not meet the given line.

Lines in the same plane that do not meet, however far extended, are said to be parallel.

**Corollary 13a.** If a line meets one of two parallel lines, it meets the other also.

For otherwise we should have two lines through a point parallel to a third line.

**Corollary 13b.** If two lines are parallel to a third line, they are parallel to each other.

For if they were not parallel they would meet, and we should have two lines through a point parallel to a third line.

All the lines parallel to a given line are parallel to each other. Together with the given line they form a system of parallels, in which every line is parallel to every other line. All lines parallel to any line of the system are members of the system.
A line that cuts a number of other lines is called a transversal of those lines.

Theorem 14. If a transversal meets two or more lines at the same angle, the lines are parallel.

\[ \text{Fig. 2} \]

GIVEN: The lines \( l, m, n \) cut by the transversal \( TV \) in the points \( H, J, K \) respectively so that \( \angle LCH = \angle LEJ = \angle LGK = \angle LBH = \angle LDJ = \angle LFK = a \). See Fig. 2.

TO PROVE: \( l, m, \) and \( n \) parallel.

PROOF: If \( l \) and \( m \) meet at \( Q \), the sum of the angles of the resulting triangle \( HJQ \) is more than 180°, which is impossible. Similarly, if \( l \) and \( m \) meet at \( R \) on the opposite side of the transversal, the sum of the angles of the resulting triangle \( HJR \) is more than 180°, which again is impossible. Since \( l \) and \( m \) cannot meet, they must be parallel.

In similar manner \( m \) and \( n \) can be proved parallel, and so \( l, m, \) and \( n \) are parallel.

NOTE: Sometimes this theorem is more easily applied if stated in the following alternative form: If a trans-
versal meets two lines so that the sum of the interior angles on the same side of the transversal is $180^\circ$, the two lines are parallel.

**Corollary 140.** Lines perpendicular to the same line are parallel.

This was proved in the first part of the proof of Theorem 13. It is also a special case of Theorem 14. These perpendiculars form a system of parallels.

**Theorem 15.** A transversal meets each line of a system of parallels at the same angle.

\[ \text{Fig. 3} \]

**GIVEN:** The parallel lines $l$, $m$, and $n$ cut by the transversal $TV$ in the points $H$, $J$, $K$ respectively. See Fig. 3.

**TO PROVE:** $LCHT = LEJT = LGKT = LBHV = LDJV = LFKV$.

**PROOF:** Construct through $J$ the line $m'$ so that $LE'JT = LCHT$. According to Theorem 14, $l$ and $m'$ must then be parallel. But $l$ and $m$ are given parallel. Therefore $m$ and $m'$ must coincide (Why?), and $LEJT$ must equal $LCHT$. 

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In similar manner we can prove that $\angle CHT = \angle GKT$. It follows immediately that $\angle BHV = \angle DJV = \angle FKV$.

**NOTE:** This theorem gives us a new and important method of proving angles equal. It enables us to prove also that the sum of two angles such as $\angle CHV$ and $\angle EJT$ in Fig. 3 is $180^\circ$.

Two angles whose sum is $180^\circ$ are sometimes called **Supplementary angles**. Two angles whose sum is $90^\circ$ are sometimes called **Complementary angles**.

**Corollary 15a.** If a line is perpendicular to one line of a system of parallels, it is perpendicular to every line of the system.

**Theorem 16.** Two transversals are cut into proportional segments by a system of parallels.

![Diagram of Theorem 16](image)

**GIVEN:** The system of parallels $AH \ldots BJ \ldots CK \ldots DL \ldots$ and the transversals $TV$ and $UW$. See Fig. 4.

**TO PROVE:** $\frac{AB}{BJ} = \frac{BC}{CK} = \frac{CD}{DL}$. 

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ANALYSIS: We can prove distances proportional by means of similar triangles. To prove triangles similar, we need equal angles, and the parallels cut by transversals supply these. Joining the transversals will provide the desired triangles.

If the given transversals happen to be parallel, as for example $TV$ and $U'W'$ (Fig. 4 on page 111), we can obtain the desired triangles by means of an auxiliary transversal $UW$.

PROOF: Let $O$ be the intersection of the transversals $TV$ and $UW$. Triangles $OAH$, $OBJ$, $OCK$, and $ODL$ have a common angle at $O$ and equal angles at $A$, $B$, $C$, $D$ (Why?). Consequently these triangles are similar,

$$\frac{OA}{OH} = \frac{OB}{OJ} = \frac{OC}{OK} = \frac{OD}{OL};$$

whence $OA = r\cdot OH$,

$$OB = r\cdot OJ,$$

$$OC = r\cdot OK,$$

$$OD = r\cdot OL.$$

If we subtract each of the first three of these equations from its successor, we obtain

$$OB - OA = r(OJ - OH),$$

or $AB = r\cdot HJ$.

Likewise, $BC = r\cdot JK$,

and $CD = r\cdot KL$.

In other words, $\frac{AB}{HJ} = \frac{BC}{JK} = \frac{CD}{KL}$.

If the given transversals $TV$ and $U'W'$ are parallel, we can use the auxiliary transversal $UW$ to prove that

$$HJ = s\cdot H'J',$$

$$JK = s\cdot J'K',$$

and $KL = s\cdot K'L'$. 

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Consequently, \( AB = rs \cdot H'J' \),
\( BC = rs \cdot J'K' \),
\( CD = rs \cdot K'L' \).
\( AB \), \( BC \), \( CD \)
and \( H'J' = J'K' = K'L' \).

**EXERCISES**

*1. In Fig. 3 on page 110 prove that \( \triangle CHT = \triangle LFKV \).

A quadrilateral the opposite sides of which are parallel is called a parallelogram.

*2. Prove that the opposite sides of a parallelogram are equal.  
*Suggestion:* A diagonal of the parallelogram may be regarded as a transversal.  
See Fig. 5.

*3. Prove that the opposite angles of a parallelogram are equal.

*4. Prove that the diagonals of a parallelogram bisect each other.  
See Fig. 6.

*5. Prove that if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.  
*Suggestion:* How can you prove two lines parallel?  Draw one diagonal of the quadrilateral and consider the two resulting triangles.

*6. Prove that if two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

7. Prove the converse of Ex. 3, namely, that if the opposite angles of a quadrilateral are equal, the quadrilateral is a parallelogram.
8. Prove the converse of Ex. 4, namely, that if the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

9. Prove that if the four sides of a parallelogram are all equal, the diagonals of the parallelogram are perpendicular.

10. Prove that if the diagonals of a parallelogram are perpendicular, the sides of the parallelogram are all equal.

11. In parallelogram $ABCD$ (Fig. 7), $AE=CG$ and $BF=DH$. Prove that quadrilateral $EFGH$ is a parallelogram.

12. In Fig. 4 on page 111, if $AB=4$, $BC=2$, $CD=3$, and $HJ=3.6$, find $JK$ and $KL$.

13. In Fig. 4, if $AB=1.4 \times CD$, what can you say about $HJ$ and $KL$?

14. In Fig. 4, if $AB=BC=CD$, what can you say about $HJ$, $JK$, and $KL$?

15. A double-tracked railroad crosses another double-tracked railroad as shown in Fig. 8. Tell all you know about the distances and angles in the figure.

16. What is the value of the product $rs$ in the last sentence of the proof of Theorem 16? What is the relation between $r$ and $s$?

*17. Draw a figure to show that the converse of Theorem 16 is not necessarily true. That is, show that if two transversals are cut into proportional segments by a system of lines, these latter lines need not be parallel.
18. Prove that if through the mid-point of one side of a triangle parallels be drawn to the other two sides of the triangle, the two smaller triangles so formed will be equal.

19. The sides of triangle $A'B'C'$ in Fig. 9 are parallel respectively to the sides of triangle $ABC$. Given that $AB = 5$, $BC = 4$, $CA = 6$, and $A'B' = 8$, find $B'C'$ and $C'A'$.

*20. Prove that if a line is parallel to one side of a triangle and intersects the other two sides, it divides these two sides proportionally (Fig. 10). Try to prove this in two ways: first, by means of similar triangles; second, by drawing another line and then using Theorem 16.

*21. Prove that if a line divides two sides of a triangle proportionally, it is parallel to the third side.

*22. Prove that the line joining the mid-points of two sides of a triangle is parallel to the third side and equal in length to half of it. See Fig. 11.

A quadrilateral having two and only two sides parallel is called a TRAPEZOID.
*23. In Fig. 12 prove that the diagonals of a trapezoid cut off proportional segments on each other.

*24. Prove that the line joining the mid-points of the nonparallel sides of a trapezoid is parallel to the parallel sides and equal in length to half their sum. Suggestion: In Fig. 13 connect \( H \), the mid-point of \( AC \), with \( E \) and \( F \). What relation does \( EH \) bear to \( BC \)? To \( AD \)? What relation does \( HF \) bear to \( AD \)? To \( BC \)? What relation does the line \( EHF \) bear to the straight line \( EF \)? Why?

*25. Prove that the bisector of an interior angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides. Suggestion: In Fig. 14, \( S \) is the point on \( AC \) such that \( AS = AB \). Through \( A \) draw a line parallel to the bisector \( BS \), and extend \( CB \) to meet it at \( R \).

*26. In Fig. 15, \( BT \) bisects the exterior angle \( ABD \) of the triangle \( ABC \). Prove that \( TA/TC = AB/BC \).

In Ex. 25 above, the point \( S \) is said to divide \( AC \) internally into the segments \( AS \) and \( SC \). In Ex. 26 the point \( T \) is said to divide \( AC \) externally into the segments \( AT \) and \( TC \). These two theorems together state that the
bisector of an interior (exterior) angle of a triangle divides the opposite side internally (externally) into segments that are proportional to the adjacent sides.

*27. Prove that the bisectors of corresponding interior and exterior angles of a triangle are perpendicular. See Fig. 16.

28. Prove that the lines joining the mid-points of the sides of a quadrilateral form a parallelogram whether or not the four vertices of the quadrilateral are in the same plane.

29. Perpendiculars are drawn to a random line from the four vertices of a parallelogram. Express the length of the perpendicular drawn to this same line from the point of intersection of the diagonals in terms of the lengths of the perpendiculars from the vertices.

Theorem 17. Perpendiculars to two perpendicular lines are themselves perpendicular.

Given: \( OA \) perpendicular to \( OB \), \( AC \) perpendicular to \( OA \), and \( BD \) perpendicular to \( OB \). See Fig. 17.

To prove: \( AC \) perpendicular to \( BD \).

Proof: We must first show that \( AC \) and \( BD \) have a point in common. \( AC \) is parallel to \( OB \), because two lines perpendicular to the same line are parallel. \( BD \), since it meets \( OB \) at \( B \), must meet \( AC \) at some point \( P \), for otherwise we should have two lines through \( B \) parallel to \( AC \).
The transversal $BD$ and the parallel lines $OB$ and $AC$ form the equal angles $OBD$ and $APD$. Consequently $AC$ and $BD$ are perpendicular. Why?

Each angle of the quadrilateral $AOBP$ is a right angle. Such a quadrilateral is called a rectangle.

Prove the following corollary.

**Corollary 17a.** The opposite sides of a rectangle are parallel and equal.

A rectangle is an equiangular parallelogram; an equilateral parallelogram is called a rhombus. A parallelogram which is both equiangular and equilateral is called a square.

**NETWORKS**

RECTANGULAR NETWORK. We have seen that all the lines perpendicular to a given line form a system of parallels. See Fig. 18. The lines perpendicular to two perpendicular lines will form two systems of parallels; by Theorem 17 every member of one system will be perpendicular to every member of the other.

The collection of lines perpendicular to two perpendicular lines is called a **rectangular network** (Fig. 19). The two given perpendicular lines are called the **axes** of the network; their intersection $O$ is called the **origin**. Every line...
perpendicular to either one of the axes belongs to the
network. Every point \( P \) in the plane lies on one and only
one line of each system of parallels in a rectangular net-
work.

Any pair of perpendicular lines each of which is per-
pendicular to an axis of the network may serve as axes of
the network, and then their intersection becomes the
origin. The network erected on the new axes will be
identical with the original network. The choice of the
particular pair of perpendicular lines which shall serve as
axes of the network is determined for the most part by
convenience and by the particular task in hand; this choice
of axes can be changed, if desired, during the course of a
demonstration.

No line not of the network can be parallel to any line
of the network; for if parallel it would then be a line of
the network.

COORDINATES. The distances \( PD \) and \( PC \) measured
along lines of the network (Fig. 19 on page 118) are the dis-
tances from point \( P \) to the axes; they are equal to \( OC \) and
\( OD \) respectively. The directed distances \( OC \) and \( OD \) locate
the point \( P \) in relation to the two axes; they are called
the COORDINATES of \( P \).

If \( OC = 5 \) and \( OD = 3 \), the coordinates of \( P \) are the num-
bers 5 and 3. The coordinates of a point are often writ-
ten after the letter designating the point, like this:
\( P: (5, 3) \). In Fig. 19 notice that the coordinates of \( P \)
and of \( P' \) are shown in this way.

The axis \( OX \) is often called the x-axis, and the axis \( OY \)
is often called the y-axis. Hence the coordinate of \( P \)
which is measured along the x-axis is called the x-coordi-
nate of \( P \); and the one which is measured along
the y-axis is called the y-coordinate of \( P \). In Fig. 19 the
x-coordinate of \( P \) is 5, and the y-coordinate is 3. What
is the z-coordinate of \( P' \)? What is the y-coordinate of \( P' \)? Notice that the z-coordinate is always written before the y-coordinate. For every point in a plane there are two numbers or coordinates which locate the point on a given network.

**The Slope of a Line.** In Fig. 20 the coordinates of the point \( P \) are \( a \) and \( b \). Notice that the relative size of \( a \) and \( b \) determines the shape of the rectangle \( CODP \) and the inclination of the line \( OP \) to the axes. The ratio \( \frac{b}{a} \) is called the slope of the line \( OP \) with respect to the \( x \)-axis. If \( a = 5 \) and \( b = 3 \), then the slope of \( OP \) is \( \frac{3}{5} \), or 0.6. We could just as well say that the slope of \( OP \) with respect to the \( y \)-axis is \( \frac{a}{b} \), but ordinarily we shall think of slopes with respect to the \( x \)-axis only.

If \( b = 0 \), the slope of \( OP \) is 0. If \( a = 0 \), we cannot speak of the slope of \( OP \) with respect to the \( x \)-axis because we cannot divide by zero.

Every line parallel to \( OP \), such as \( QR \) in Fig. 20, forms an angle of the same size with the \( x \)-axis as \( OP \) forms. This means that the right triangles \( QRE \) and \( OPC \) are
similar and that the slope of \( QR \) can be written either as \( \frac{b}{a} \), taking \( O \) as origin, or as \( \frac{a}{b} \). We see, therefore, that every line parallel to \( OP \) has the same slope as \( OP \).

EXERCISES

1. Locate the following points on a rectangular network, taking one-quarter of an inch or any other convenient distance as the unit of measurement: \( A:(3, 5); \) \( B:(4, 2); \) \( C:(2, 3); \) \( D:(6.50, 2.75). \)

2. Draw a line through each of these points and the origin and find its slope.

3. Locate each of the following points on your network: \( E:(-5, 3); \) \( F:(5, -3); \) \( G:(-5, -3); \) \( H:(-4, 2); \) \( I:(2, -5); \) \( J:(-2, -3.5); \) \( K:(-3.5, -2.5). \)

4. Find the distance from the origin to each of the points mentioned in Exercises 1 and 3. (Use the Pythagorean Theorem.)

5. Compute the following distances: \( AC, BD, AG, EF, EK, CJ, JK, ID, KD, JD. \) (Use the Pythagorean Theorem.)

6. Draw a line through the origin and the point \( H \) and find its slope. Notice that one of the coordinates of \( H \) is negative.

7. Draw the lines \( CA, EH, \) and \( HF \) and find the slope of each of these three lines.

8. Compare the slopes of lines \( GK \) and \( BD. \) What is the slope of line \( HB? \)

9. Pick out the line segment on your network that has the steepest slope. Pick out a line segment on your network that has no slope.
Theorem 18. If two lines have a point in common and have equal slopes, they must coincide.

GIVEN: Lines PQ and PR having equal slopes and passing through P. See Fig. 21.

TO PROVE: PQ and PR must coincide.

ANALYSIS: To prove that these lines coincide, we must show that they make the same angle with some line of the network. We can prove angles equal by means of the three Cases of Similarity; the equal slopes suggest sides of right triangles in proportion and the advisability of trying to apply Principle 5.

PROOF: If point P is not at the origin of the network to which the slopes are referred, we have only to choose P as a new origin and choose for new axes the lines of the network which pass through P. Complete the proof.

THE EQUATION OF A LINE. The slope of the line through the origin 0 and the point P is given in terms of the coordinates of P. In Fig. 22 on the next page the slope of OP is \( \frac{b}{a} \). But the slope of this line can be stated equally well in terms of the coordinates \((x, y)\) of any other
point on the line, such as the point \( V \). That is, the slope of \( DV \) is equal to the slope of \( OP \); for triangles \( LOP \) and \( MOV \) are similar, by Case 2 of Similarity (Principle 6). It follows that \( \frac{y}{x} = \frac{b}{a} \). This equation is called the equation of the line \( OP \). Any point except 0 on \( OP \) must have coordinates \((x, y)\) which satisfy this equation, and by Theorem 18 any point whose coordinates \((x, y)\) satisfy this equation must lie on \( OP \). Thus the equation \( \frac{y}{x} = \frac{b}{a} \) is an algebraic way of saying that the ratios of the coordinates of all points except \((0, 0)\) on line \( OP \) are equal.

If the point 0 does not happen to be at the origin, it is always possible to choose a new pair of axes for the network so that the origin shall fall at O.

The equations of lines that are members of the network can be obtained directly without using slopes. For example, in Fig. 23 notice the line of the network that is parallel to the \( x\)-axis and passes through the point \((a, b)\). The \( y\)-coordinate of every point on this line is \( b \); therefore the equation of this line is \( y = b \). Similarly, since the \( x\)-coordinate of every point on the line through \((a, b)\) perpendicular to the \( x\)-axis is \( a \), the equation of this line is \( x = a \). The equation of the line forming the \( x\)-axis is \( y = 0 \); the equation of the line forming the \( y\)-axis is \( x = 0 \).
EXERCISES

1. Write the equation of the line through the origin and the point (5, 4).

2. Write the equation of the line through the origin and the point (-2, 3).

3. Rewrite the equation \( \frac{y}{x} = \frac{b}{a} \) so that it will be satisfied by the coordinates of the origin (0, 0) and at the same time avoid having a fraction with a zero denominator.

SYMMETRY. Two points, \( P \) and \( P' \), are said to be symmetric with respect to a line if the line is the perpendicular bisector of \( PP' \). The line is called the axis of symmetry.

Notice what happens when we fold the geometric figure shown in Fig. 24 along the line \( AB \). One-half of the figure then coincides with the other half. We say that this figure is symmetric with respect to the axis \( AB \).

A geometric figure is symmetric with respect to an axis if every point \( P \) of the figure (except points on the axis) has a corresponding point \( P' \) in the figure such that \( PP' \) is bisected perpendicularly by the axis.

An equilateral triangle has three axes of symmetry. How many has a square? A rectangle? A regular pentagon? A regular hexagon?

Show by folding a figure that if two lines intersect, the axes of symmetry of the figure bisect the angles between the lines.

An equilateral triangle is divided by an axis of symmetry...
into two right triangles each of which is said to be symmetric to the other.

In three dimensions we can have a plane of symmetry analogous to the axis of symmetry in two dimensions.

How many planes of symmetry has a brick? A cube? A man? Is there any feature in which two symmetric triangles on a sphere differ from two symmetric triangles on a plane?

Figure 25 has no axis of symmetry. If, however, we rotate this figure in the plane of the paper about the point 0, it coincides from time to time with its original position. We say that this figure is symmetric with respect to the point O. The point 0 is called the center of symmetry of the figure.


Bring to class pictures of buildings, church windows, formal designs, and window tracery which show various kinds of symmetry. Look for examples of symmetry in nature. Look in a dictionary, an encyclopedia, or a book on trees for pictures of mulberry and sassafras leaves. If you can find pictures of twigs with several leaves on them, notice in what ways they illustrate symmetry and in what ways they illustrate lack of symmetry. Copy the pictures to show to the rest of the class.

Using as a pattern the definition of symmetry with respect to an axis given on page 124, state the definition of symmetry with respect to a point in such a way that it will apply to geometric figures with complicated boundaries, such as the block letter S shown in Fig. 26 on page 126.

Three or more points are said to be collinear if they all lie on the same straight line. We sometimes say that such
points have a line in common. *Three or more lines are said to be concurrent if they all pass through the same point.* We sometimes say that such lines have a point in common.

**EXERCISES**

1. Designers of ornaments make frequent use of axial symmetry, central symmetry, and the close relation of a figure to a network. How many axes of symmetry has each of the seven figures below? Which ones have symmetry with respect to a point? Which ones show close relation to a network?

   ![Fig. 26](image)

2. The leaf pictured in Fig. 27 is from poison-ivy. What is there about the shape of poison-ivy leaves that would help you identify this plant outdoors?

   ![Fig. 27](image)

*3. Prove that if three or more lines are concurrent, they cut off proportional segments on two parallel lines. See Fig. 28 on the next page.*
4. Prove that if three lines cut off proportional segments on two parallel lines, they are either parallel or concurrent. *Suggestion:* In Fig. 29 assume that $AA'$ and $BB'$ meet at $P$ and that $BB'$ and $CC'$ meet at $Q$. In order to prove that $P$ and $Q$ coincide, you must do more than prove that $\frac{PB'}{OB'} = \frac{QB'}{OB'}$. You must go further and prove that $\frac{PB}{BB'} = \frac{QB}{BB'}$. In this connection look again at Exercises 20-38, pages 64-66.

5. Prove that if two triangles have their sides respectively parallel, they are similar.

6. Prove that if two similar triangles have their sides respectively parallel (Fig. 30), the lines $AA'$, $BB'$, $CC'$ joining corresponding vertices are concurrent. This point of concurrence is sometimes called the "center of similarity" of two triangles.
7. There is one exception to the proposition in Ex. 6 on page 127. What is it? What about $AA'$ and $BB'$ in this case? $AA'$ and $CC'$?

8. Prove the proposition in Ex. 6 when the point of concurrence lies between the triangles. See Fig. 31.

9. Two lines $l$ and $m$ (Fig. 32) intersect at an inaccessible point. It is desired to draw through a given point $P$ a line $m$ which if prolonged would pass through the intersection of $l$ and $m$. Show how this can be done, assuming that you have a device for drawing a line parallel to a given line.

10. While walking north on Main Street, Fred Harris spies John Reed driving south in a car at 15 miles an hour, just before an elm tree near the curb hides the car from his view. Fred does not want John to see him. How fast must Fred continue walking north in order to keep hidden by the elm tree, if the tree is four times as far from John as from Fred?

11. Prove that the perpendicular bisectors of the sides of a rectangle meet at the point of intersection of the diagonals.

12. Three planes ordinarily intersect in a point, as, for example, the floor and two adjacent walls of a room. Under other conditions three planes may have a line in common. Explain the conditions under which three planes can intersect (two at a time) in three lines; in two lines. When will the planes have no point in common?

In Exercises 13-20 the word "Show" is used instead of "Prove" because in these exercises you are expected principally to see the relations between the points, lines,
and planes involved and are not required to give the care­ful sort of explanation and justification ordinarily ex­pected of you.

13. Show that if two lines are parallel, every plane con­taining one of the lines, and only one, is parallel to the other. *Suggestion:* Use the indirect method.

14. Show that two planes perpendicular to the same line must be parallel. *Suggestion:* Suppose they have a point in common.

15. Show that if two parallel planes are cut by a third plane, the lines of intersection are parallel. *Suggestion:* Use the indirect method.

16. Show that if two lines are cut by three parallel planes, their corresponding segments are proportional. See Fig. 33.

17. Show that if a pyramid is cut by a plane parallel to the base, the lateral edges and the altitude are divided proportionally, and the section is a polygon similar to the base. See Fig. 34.
18. Show that if each of two intersecting lines is parallel to a given plane, the plane of these lines is parallel to the given plane. *Suggestion:* Use the indirect method.

Two lines that do not meet and are not in the same plane are called **skew lines**.

19. Show that through either of two skew lines it is possible to pass one plane, and only one, which is parallel to the other line. *Suggestion:* Through any point of one of the given lines draw a line parallel to the other given line.

20. Show that through a given point in space one plane, and only one, can be passed parallel to each of two skew lines, or else parallel to one line and containing the other.

21. See if you can convince yourself that between any two given skew lines there is one common perpendicular, and only one. *Suggestion:* Consider: first a random plane parallel to the two given skew lines. Then consider the relation of this plane to all lines that are perpendicular to either of the two skew lines.

**SUMMARY**

13. Through a given point not on a given line there is one and only one line which does not meet the given line.

**DEFINITION:** *parallel lines*

13a. If a line meets one of two parallel lines, it meets the other also.

13b. If two lines are parallel to a third line, they are parallel to each other.

**DEFINITIONS:** *system of parallels, transversal*
14. If a transversal meets two or more lines at the same angle, the lines are parallel.

14a. Lines perpendicular to the same line are parallel.

15. A transversal meets each line of a system of parallels at the same angle.

DEFINITIONS: supplementary angles, complementary angles

15a. If a line is perpendicular to one line of a system of parallels, it is perpendicular to every line of the system.

16. Two transversals are cut into proportional segments by a system of parallels.

DEFINITIONS: parallelogram, trapezoid

17. Perpendiculars to two perpendicular lines are themselves perpendicular.

DEFINITION: rectangle

17a. The opposite sides of a rectangle are parallel and equal.

DEFINITIONS: rhombus, square; rectangular network, axes, origin, coordinates, slope of a line

18. If two lines have a point in common and have equal slopes, they must coincide.

DEFINITIONS: equation of a line; symmetry with respect to a line, axis of symmetry, symmetry with respect to a point, center of symmetry; collinear points, concurrent lines
The building shown above is Westminster Abbey, the most famous church in England. Much of the beauty of this building is derived from geometric figures. The insert shows a design based on tangent circles which is used on the front of the building.

The picture below shows a football team attempting to score by a kick. The accompanying diagram shows that the angle between the goal posts is the same whether the kicker is near the side of the field on the 20-yard line or midway between the sides of the field on the 40-yard line. This diagram illustrates the theorem that all inscribed angles having the same arc are equal.
CHAPTER 5

The Circle and Regular Polygons

IN THE PRECEDING CHAPTERS we have con­sidered only straight lines and figures made up of straight lines. We turn now to the simplest of all curved lines, the circle. We shall study its properties and its relation to straight lines and to figures made up of straight lines, especially polygons.

In a plane all the points at a given distance from a given fixed point are said to form a circle.

The fixed point 0 is called the CENTER of the circle, and the distance \( r \) is called the RADIUS (Fig. 1).

Every point at a distance \( r \) from 0 is said to be on the circle. Every point at a distance less than \( r \) from 0 is said to be inside the circle, and every point at a distance greater than \( r \) from 0 is said to be outside the circle.

If the center of the circle be taken as the origin of a rectangular network, it follows from the Pythagorean Theorem (Principle 12) that the coordinates \((x, y)\) of every point \( P \) of the circle will satisfy the equation \( x^2 + y^2 = r^2 \). This equation is the EQUATION OF THE CIRCLE. Every circle with center 0 has an equation of this sort, with a different value of \( r \) for each circle.
On any half-line with endpoint 0 there is a point at the distance \( r \) from O. This point is on the circle with center 0 and radius \( r \). We may select one such half-line—for example, OX in Fig. 2—as a reference line from which to measure the angles to all other such half-lines.

If we measure these angles in degrees, then on every half-line which makes an angle of between 0 and 360 degrees with OX there is a point of the circle.

All the points of the circle which lie on half-lines bearing numbers from \( p \) to \( q^* \) are said to form an arc \( PQ \) of the circle. The word "arc" comes from a Latin word meaning "bow." In Fig. 3 arc \( PQ \) corresponds to angle \( POQ \). Angle \( POQ \) is called a central angle because its vertex is at the center of the circle.

In Fig. 3 notice that the half-lines \( OP \) and \( OQ \) form two angles whose sum is 360°. Ordinarily when we speak of angle \( POQ \) we refer to the lesser of these two angles; only rarely do we mean the greater angle. Similarly, when we speak of the arc \( PQ \), we ordinarily mean the arc that corresponds to the lesser central angle \( POQ \); but occasionally we mean the arc that corresponds to the greater central angle. Except for the end-points \( P \) and \( Q \), all the points of the first arc \( PQ \)-sometimes called the minor arc—are distinct from the points of the second arc \( PQ \)-sometimes called the major arc. If the two central angles \( POQ \) are equal, each of the two corresponding arcs \( PQ \) is called a semicircle.

*Or, in certain cases, from \( p \) to \( 0 \) and from 360 to \( q \); or from \( p \) to 360 and from 0 to \( q \).
For convenience we shall refer to two arcs as **EQUA\_\_L ARCS** if their circles have equal radii and if their central angles are equal. It is obvious geometrically that equal arcs have equal lengths, but we shall postpone till Chapter 7 all questions relating to lengths of curves.

In Fig. 4 a point P of the arc \( AB \) is said to bisect the arc if the central angles \( AOP \) and \( POB \) are equal. P is called the **MID-POINT OF THE ARC**.

Circles which have equal radii are called **EQUA\_\_L CIRCLES**.

**EXERCISES**

1. The circles whose radii are given below have their centers at the origin of a rectangular network. Write the equation of each circle.
   - (a) 2
   - (b) 5
   - (c) 3.1
   - (d) \( \frac{5}{3} \)

2. Find the radius of each circle whose equation is given below.
   - (a) \( x^2 + y^2 = 9 \)
   - (b) \( x^2 + y^2 = 8 \)
   - (c) \( x^2 + y^2 = 3.61 \)
   - (d) \( 4x^2 + 4y^2 = 9 \)

3. How many points are there whose coordinates satisfy the equation \( x^2 + y^2 = 0 \)?

4. How many points are there whose coordinates satisfy the equation \( x^2 + y^2 = -4 \)?

5. Pick out all the pairs of equal arcs you can find in Fig. 5.

6. How would you locate the point bisecting arc \( BE \) in Fig. 5?

*The word "equal" here is taken to mean "equal in all respects," or "congruent." See page 59.*
A straight line segment joining two points of a circle is called a chord. See Fig. 6.

A chord which passes through the center of the circle is equal in length to twice the radius; this length is called the diameter of the circle. The word "chord" comes from the Greek word for "string." Look in the dictionary for the origin of the words "radius" and "diameter." As indicated on page 15, we shall use the words "radius" and "diameter" to denote certain lines as well as their lengths, when to do so will cause no confusion.

Now let us see just how the chords of a circle may vary in length. A chord of a circle can never be longer than $2r$, although it may be equal to $2r$. What do we call a chord that is equal to $2r$? No chord can be so short that its length is equal to zero. The two points of a circle which form the end-points of a chord may be as near as we like, making the chord as short as we please. No matter how small a number you name, we can always find a chord whose length is less than the number; but we cannot find any chord whose length is zero.

**Theorem** 19. In the same circle, or in equal circles, equal chords have equal arcs; and conversely.

(1) **GIVEN:** Circle $O$ (Fig. 7) in which $AB = CD$.

TO PROVE: $\text{Arc } AB = \text{arc } CD$.

Analyze and prove this theorem. 
*Suggestion:* Use Case 3 of Similarity (Principle 8) and the definition of equal arcs.

(2) **GIVEN:** Circle $O$ (Fig. 7), in which arc $AB = \text{arc } CD$.

TO PROVE: $AB = CD$. 

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Now analyze and prove the converse of the theorem.  
* Suggestion: Use the definition of equal arcs and Principle 5.

**EXERCISES**

*1. On a circle there are several points, A, B, C, D, E, ... equally spaced. Prove that AC, BD, CE, ... are equal chords. Prove also that AD, BE, ... are equal.

*2. Prove that the line joining the center of a circle to the mid-point of a chord is perpendicular to the chord.

*3. Prove that a diameter perpendicular to a chord of a circle bisects the chord and the corresponding arcs.

*4. Prove that in the same circle (Fig. 8), or in equal circles, equal chords are equidistant from the center.  * Suggestion: Use the Pythagorean Theorem.

*5. Prove that in the same circle, or in equal circles, chords equidistant from the center are equal.

*6. Prove that if in the same circle, or in equal circles, two chords are unequal, the shorter is at the greater distance from the center.

7. Successive arcs are marked off on a circle so that each arc has a chord equal in length to the radius. Prove that the sixth arc ends at the point where the first arc begins.

8. Prove that six dimes will just fit around the rim of a seventh dime so that each of the six outside dimes will just touch three other dimes.  * Suggestion: Connect the centers of neighboring dimes by straight lines.

* An endless straight line passing through two points of a circle is called a secant. Notice that every chord is a particular segment of a secant.
The secant can have only two points in common with the circle. In Fig. 9 these points are \( A \) and \( B \). If there were a third point \( C \) on the secant at a distance \( r \) from \( O \), then \((DC)^2\) would equal \( r^2 \). \((OD)^2\), where \( D \) is the foot of the perpendicular from \( O \). But \((AD)^2 = r^2 - (OD)^2\). Therefore \((DC)^2\) must equal both \((DB)^2\) and \((DA)^2\), and hence \( C \) is not distinct from both \( A \) and \( B \) (by Principle 1 and the second paragraph on page 43).

When a line and a circle have two points in common, we say that the line and the circle intersect. The points they have in common are called the points of intersection.

A straight line having but one point in common with a circle is called a tangent.\(^*\) See Fig. 10. The common point is called the point of tangency, or point of contact. The line is tangent to the circle, and the circle is tangent to the line, at the point of tangency.

If the sides of a polygon are chords of a circle, the polygon is said to be inscribed in the circle; the circle is said to be circumscribed about the polygon.

If the sides of a polygon are tangent to a circle, the polygon is said to be circumscribed about the circle; the circle in this case is said to be inscribed in the polygon. See Fig. 11.

\(^*\)It is understood, of course, that the line lies in the plane of the circle.
**Theorem 20.** A line perpendicular to a radius at its outer extremity is tangent to the circle.

**GIVEN:** Circle 0, radius $OT$, and line $l$ perpendicular to $OT$ at $T$. See Fig. 12.

**TO PROVE:** Line $l$ has but one point in common with the circle.

**PROOF:** Let $U$ be any point on $l$ other than $T$. Then $(OU)^2 = (OT)^2 + (TU)^2$. This means that $OU > OT$ and that $U$ is outside the circle. Therefore every point of line $l$ except $T$ is outside the circle, and $l$ has but one point, $T$, in common with the circle. In other words, line $l$ is tangent to the circle.

**Theorem 21.** Every tangent to a circle is perpendicular to the radius drawn to the point of contact.

**GIVEN:** Circle 0 and line $l$ tangent to the circle at $T$. See Fig. 13.

**TO PROVE:** Line $l$ is perpendicular to $OT$.

**ANALYSIS:** In proving Theorem 20 we used the given perpendicular relation and the Pythagorean Theorem to show that every point on $l$ except $T$ was outside the circle and therefore that $l$ had only one point in common with the circle. In proving the converse theorem we may be able to reverse these steps. Starting with the given idea, only one point in common with the circle, we must try to prove that every point on $l$ except $T$ is out-
side the circle, and so prove that $OT$ is perpendicular to $l$. In proving this last step we cannot use the Pythagorean Theorem directly because now the perpendicular relation is in doubt. But there is a corollary of the Pythagorean Theorem that connects perpendicularity with the idea of shortest distance from a point to a line, and we may be able to use that here. Evidently then the proof depends upon our ability to show that $T$ is the nearest point of line $l$ to $O$. We can do this by the indirect method, showing that if $T$ were not nearer to $O$ than any other point of $l$ we should have a contradiction.

**Proof:** Let $U$ be any point of $l$ other than $T$.

1. $OU$ cannot equal $OT$, for then $l$ would have two points in common with the circle.

2. $OU$ cannot be less than $OT$, for if it were, some point of $l$ other than $T$ would be nearest to $O$. Call this nearest point $Q$. Then $OQ$ would be perpendicular to $l$ (Corollary 12d), and there would be a second point $T'$ on $l$ at a distance $QT$ from $Q$ (page 43). This would make $OT'$ equal to $OT$, and $l$ would have two points in common with the circle, which is impossible.

3. Therefore $OU$ must be greater than $OT$, and $OT$ is the shortest distance from $O$ to $l$. It follows from Corollary 12d that $OT$ and $l$ are perpendicular.

**Corollary 21a.** There is only one tangent to a circle at any given point of the circle.

For otherwise there would be at least two distinct perpendiculars to $OP$ at $P$, which is impossible.

**Exercises**

*1. Prove that a perpendicular to a tangent at the point of tangency passes through the center of the circle.*
2. Prove that a perpendicular from the center of a circle to a tangent passes through the point of tangency.

In Fig. 14 arcs $AC$ and $BD$ lie between lines $AB$ and $CD$. We say that lines $AB$ and $CD$ intercept arcs $AC$ and $BD$.

![Fig. 14](image)

![Fig. 15](image)

3. Prove that two parallel secants intercept equal arcs on a circle. Suggestion: In Fig. 15 draw $MOM'$ perpendicular to $AB$ and show that $LAOC$ is equal to $LBOD$.

4. Prove that a secant and a tangent parallel to it intercept equal arcs on a circle.

5. Prove that two parallel tangents intercept equal arcs on a circle.

6. Prove that every parallelogram inscribed in a circle is a rectangle.

7. Prove that every trapezoid inscribed in a circle is isosceles.

8. Prove that if two tangents to a circle intersect, they make equal angles with the line joining the point of intersection to the center, and their segments from the point of intersection to the points of tangency are equal. See Fig. 16.

![Fig. 16](image)
9. In Fig. 17 a quadrilateral is circumscribed about a circle. Prove that the sum of one pair of opposite sides is equal to the sum of the other pair.

10. Draw lines from the vertices of the quadrilateral in Fig. 17 to the center of the circle and see if you can discover a relation between the central angles. Prove it.

11. In Fig. 18 a hexagon $ABCDEF$ is circumscribed about a circle. Prove that $AB + CD + EF = BC + DE + FA$.

12. Could you prove a theorem similar to the one in Ex. 11 for a polygon of eight sides circumscribed about the circle? For a polygon of ten sides? For a polygon of eleven sides?

State your conclusions in as general terms as possible.

The straight line that contains the centers 0 and $0'$ of two circles is called the line of centers of the two circles.

13. Draw diagrams to show that if the straight line segment $00'$ joining the centers 0 and $0'$ of two circles is greater than the sum of the radii, or less than the difference of the radii, the circles have no point in common. Show similarly that if $00'$ is equal to the sum of the radii, or to their difference, the two circles have one and only one point in common. This common point must lie on $00'$ or on $00'$ extended, for otherwise Corollary 12c would be violated. Show finally that if $00'$ is less than the sum and greater than the difference of the radii, the circles have at least two points in common. In no case of this sort can a point common to the two circles lie on $00'$ or
on 00’ extended; and corresponding to each common point P there must be a second point Q such that P and Q are symmetric with respect to the line 00’.

*14. Prove that if two circles have two points in common, the line of centers is the perpendicular bisector of the line segment joining these two points.

*15. Prove that two circles cannot have more than two points in common. Suggestion: Use the indirect method and Ex. 14.

When two circles have two points in common, we say that the two circles intersect. The points that they have in common are called the points of intersection, and the line segment joining these points is called the common chord of the two circles.

If two circles are tangent to the same line at the same point, the circles are said to be tangent to each other.

*16. Prove that if two circles are tangent to each other, the line of centers passes through the point of contact (Fig. 19).

If the point of contact of two tangent circles lies between the centers, the circles are said to be tangent externally; if the point of contact does not lie between the centers, the circles are said to be tangent internally.

In Fig. 19 which two circles are tangent externally? Which two circles are tangent internally?
A line which is tangent to two circles is called a COMMON INTERNAL TANGENT if it has a point in common with the line segment joining the two centers; if it has not, it is called a COMMON EXTERNAL TANGENT. See Fig. 20.

Circles that are tangent internally have a common external tangent; whereas circles that are tangent externally have a common internal tangent and two common external tangents.

17. Illustrate by diagrams the relative positions of two circles which satisfy in turn each of the following conditions.

(a) Having two common external tangents and two common internal tangents.

(b) Having two common external tangents and one common internal tangent.

(c) Having two common external tangents and no common internal tangent.

(d) Having one common external tangent and no common internal tangent.

(e) Having no common tangent.

18. Two circles are tangent externally. Their diameters are 4.86 inches and 3.12 inches. Find the distance between the centers of the circles.

19. Two circles having the same diameters as those in Ex. 18 are tangent internally. Find the distance between the centers of these circles.

20. Three circles with diameters of 5.2 inches, 3.6 inches, and 4.4 inches are externally tangent, each to the other two. Find the perimeter of the triangle formed by joining the centers.
21. For each of the cases in Ex. 17 state a relation which must hold between the length of the line segment joining the centers and the sum or difference of the radii.

22. What is the axis of symmetry in each case in Ex. 17?

23. Prove that if two circles are tangent either externally or internally at \( T \), tangents to both circles from any point except \( T \) of the common tangent are equal.

24. Prove that if two circles are tangent externally, their common internal tangent bisects both common external tangents.

25. Prove that the common external tangents of two circles meet on the line of centers.

26. Prove that the common internal tangents of two circles meet on the line of centers.

An angle whose vertex is on a circle, and whose sides are chords, is called an **INSCRIBED ANGLE**.

*Theorem 22.* An inscribed angle is equal to half the central angle having the same arc.

![Diagram](image)

**CASE 1**

**CASE 2**

**CASE 3**

**Fig. 21**

**GIVEN:** Angle \( \angle ABC \) inscribed in a circle with center \( O \); central angle \( \angle ADC \). See Fig. 21.

**TO PROVE:** \( \angle ABC = \frac{1}{2} \angle AOC \).
Case 1. When one side of the inscribed angle goes through O.

**Proof:** \( \angle ABC + \angle BCO + \angle COB = 180^\circ \) (By Theorem 9), and \( \angle AOC + \angle COB = 180^\circ \). (Why?)

Therefore \( \angle ABC + \angle BCO = \angle AOC \).

But \( \angle ABC = \angle BCO \). (Why?)

Therefore \( 2\angle ABC = \angle AOC \),

and \( \angle ABC = \frac{1}{2}\angle AOC \).

Case 2. When \( O \) lies within the angle \( ABC \). *

**Proof:** Draw through \( B \) and \( O \) the chord \( BD \). This chord divides \( ABC \) into two parts each of which falls under Case 1. Now complete the proof.

Case 3. When \( O \) lies outside the angle \( ABC \).*

**Proof:** Draw through \( B \) and \( O \) the chord \( BD \). Complete the proof.

**Corollary 22a.** Equal angles inscribed in the same circle have equal arcs.

**Corollary 22b.** All inscribed angles having the same arc are equal. See Fig. 22.

**Corollary 22c.** Every angle “inscribed in a semicircle” is a right angle. See Fig. 23.

*The expressions “within the angle” and “outside the angle” can be defined precisely by means of the idea of “betweenness,” but their meaning here is obvious from the diagrams in Fig. 21 on page 145.
EXERCISES

1. Prove that if a quadrilateral has its vertices lying in a circle, its opposite angles are supplementary (that is, add up to 180°).

2. Prove that an inscribed triangle is a right triangle if one side passes through the center of the circle.

3. Through one of the points of intersection of two circles chords are drawn through the centers of the circles. Prove that the line joining the ends of these chords passes through the other point of intersection of the circles. *Suggestion:* In Fig. 24 prove that \(CBD\) is a straight line.

4. Through each of the two points of intersection of two circles a line is drawn. The two lines so drawn are terminated by the circles. Prove that the chords joining the corresponding ends of the two lines are parallel. Prove this for both cases shown in the diagrams in Fig. 25.

*5. Prove that an angle formed by two intersecting chords of a circle is equal to half the sum of the central angles having the same arcs as the given angle and the equal angle opposite it. *Suggestion:* Draw \(CB\) (Fig. 26) and prove first that \(\angle APC\) is equal to the sum of two inscribed angles.
6. Prove that an angle formed by two secants intersecting outside a circle is equal to half the difference of the central angles corresponding to the arcs intercepted by the secants. Suggestion: Draw $CB$ (Fig. 27) and prove first that $\angle APC$ is equal to the difference of two inscribed angles.

7. Prove that an angle formed by a tangent and a chord of a circle (Fig. 28) is equal to half the central angle having the same arc as the chord.

8. Prove that an angle formed by a tangent and a secant (Fig. 29) is equal to half the difference of the central angles corresponding to the intercepted arcs. Suggestion: Make use of Ex. 7.

9. Prove that an angle formed by two intersecting tangents is equal to $180^\circ$ minus the central angle having the same minor arc. See Fig. 30.

10. Prove that the angle formed by two intersecting tangents can be expressed in the same way as the angle between two secants that intersect outside a circle, that is, as equal to half the difference of the central angles corresponding to the intercepted arcs.
11. If in Fig. 26 on page 147 arcs $AC$ and $BD$ have central angles of $71°$ and $56°$ respectively, how many degrees has angle $APC$?

12. If in Fig. 26 arcs $BC$ and $AD$ have central angles of $203°$ and $38°$ respectively, how many degrees has angle $APC$?

13. If in Fig. 26 $\angle APC = 64°$ and arc $BD$ has a central angle of $59°$, how many degrees has the central angle of arc $AC$?

14. If in Fig. 26 $D$, and consequently also $P$, coincides with $B$, show that the inscribed angle $ABC$ can be regarded as a limiting case of the angle between two intersecting chords.

15. If in Fig. 27 on page 148 arcs $AC$ and $BD$ have central angles of $94°$ and $39°$ respectively, how many degrees has angle $APC$?

16. If in Fig. 27 $\angle APC = 29°$ and arc $AC$ has a central angle of $92°$, how large is the central angle of arc $BD$?

17. If in Fig. 27 $D$, and consequently also $P$, coincides with $B$, show that the inscribed angle $ABC$ can be regarded as a limiting case of the angle between two secants that intersect outside a circle.

18. Test the theorem of Ex. 7, page 148, by the case in which the angle $TOB$ is $90°$; by the case in which it is $180°$.

19. If in Fig. 29 on page 148 arcs $TA$ and $TB$ have central angles of $76°$ and $51°$ respectively, how many degrees has angle $TPA$?

20. If in Fig. 29 $\angle TPA = 13°$ and arc $TA$ has a central angle of $78°$, how large is the central angle of arc $TB$?

21. If in Fig. 30 on page 148 the major arc $ST$ has a central angle of $223°$, how many degrees has angle $SPT$?
22. If in Fig. 30 $\angle 8PT = 42.2^\circ$, how large are the central angles of the major and minor arcs $8T$?

23. Show that the theorem in Ex. 8, page 148, can be regarded as a limiting case of the theorem in Ex. 6 on the same page.

24. Show that the theorem in Ex. 7, page 148, can be regarded as a special case of the theorem in Ex. 8 on the same page.

25. Show that the theorem in Ex. 10, page 148, can be regarded as a limiting case of the theorem in Ex. 8 on the same page.

26. In Fig. 31 prove that the perpendicular $PD$ from any point $P$ on a circle to a diameter* $AB$ is the mean proportional between the segments of the diameter. Suggestion: Use Corollary 22c and Ex. 14, page 85.

27. Chords $PA$ and $PB$ are drawn from any point $P$ on a circle to the extremities of a diameter $AB$, and $D$ is the foot of the perpendicular from $P$ to $AB$. Prove that $(PA)^2 = AD \times AB$ and $(PB)^2 = DB \times AB$.

28. If in Fig. 31 $AD = 2$ and $AO = 4$, find $PD$.

29. If in Fig. 31 $PD = 3$ and $AD = 2$, find $AO$.

30. If in Fig. 31 $PD = 3$ and $PO = 4$, find $AD$.

31. If in Fig. 31 $AD = 3$ and $DB = 5$, find $PA$ and $PB$.

32. If in Fig. 31 $AD = 2$ and $AO = 4$, find $PA$ and $PB$.

33. If in Fig. 31 $PA = 3$ and $AD = 1.5$, find $AB$ and $PB$.

34. If in Fig. 31 $PA = 4$ and $PB = 6$, find $AD$ and $PD$.

*The word "diameter" here means the chord through the center of the circle, rather than its length. The word is frequently used in this sense when there is no cause for misunderstanding. The word "radius" is often used in a similar manner to denote the line rather than its length. On the other hand, we use "perpendicular" here to denote the length rather than the line.
*35. Prove that if two chords in a circle intersect, the product of the segments of one is equal to the product of the segments of the other. *Suggestion:* Show that triangles $PAC$ and $PDB$ in Fig. 32 are similar.

\[ AC \parallel PA \parallel PC \]

\[ \frac{PA}{PD} = \frac{PC}{PB} \]

*36. Prove that in Fig. 33 $\frac{AC}{DB} = \frac{PA}{PD} = \frac{PC}{PB}$ and, consequently, that $PA \times PB = PD \times PC$.

*37. Prove that if from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment. See Fig. 34.

In cases of this sort we find it convenient to refer to the lengths $PT$ and $PB$ in Fig. 34 by the abbreviated expressions "tangent" and "secant" respectively, instead of using much longer expressions to describe the lengths of these particular line segments.

*38. Prove that if from a fixed point outside a circle a secant is drawn, the product of the secant times its external segment is constant in whatever direction the secant is drawn.

39. If in Fig. 32 $AP=4$, $PB=5$, and $CP=7$, find $PD$.

40. If in Fig. 32 $AP=3.2$, $AB=7.1$, and $PD=2.6$, find $CD$.

41. If in Fig. 33 $AP=11$, $BP=7$, and $CP=6$, find $DP$. 
42. If in Fig. 33 $AB = 4$, $BP = 8$, and $DC = 5$, find $CP$.

43. If in Fig. 34 $BA = 4$ and $AP = 6$, find $TP$.

44. If in Fig. 34 $TP = 8$ and $BA = 4$, find $AP$.

45. If in Fig. 34 $TP = 7.5$, what will be the product of any secant times its external segment?

46. Prove that if two lines $AB$ and $CD$ intersect a circle in the four points $A$, $B$, $C$, and $D$ and have point $P$ in common, then $PA \times PB = PC \times PD$ whether $P$ be inside or outside the circle.

The theorem in Ex. 46 summarizes the theorems of Exercises 35, 36, and 38 on page 151. They may be generalized in somewhat different language as follows: The product of the distances from a given point to two points lying on a circle and collinear with the given point is independent of the direction of the joining line.

47. Show that the equation in Ex. 46 can be regarded as true even in the limiting case when one of the secants becomes a tangent; also when both secants become tangents.

Theorem 23 concerns regular polygons. These were defined on page 85.

Theorem 23. A circle can be circumscribed about any regular polygon.

**GIVEN:** Regular polygon $ABCD$ .... See Fig. 35.

**TO PROVE:** A circle can be circumscribed about $ABCD$ ....

**ANALYSIS:** The main problem is to find the center of a circle which passes through $A$, $B$, $C$, $D$, .... That is, we must find a point which is equidistant from $A$, $B$, $C$, $D$, .... Let us see first of all if we can find a point which is equidistant from
A and B; then see if we can find a point equidistant from A, B, and C; from A, B, C, and D; and so forth.

**PROOF:** We already know (by Principle 10) that all the points equidistant from A and B lie on the perpendicular bisector PM of AB. Any point of PM can serve as center of a circle through A and B; and similarly, if QN is the perpendicular bisector of BC, any point of QN can serve as center of a circle through B and C. Therefore 0, the point of intersection of PM and QN, will be the center of a circle through A, B, and C. We know that PM and QN must intersect: for if they were parallel, each would be perpendicular to both AB and BC (by Corollary 15a), and we should have two lines through B perpendicular to the same line, which is impossible (by Principle 11). This assumes, of course, that AB and BC lie on different lines; that is clearly so because angle ABC, at a vertex of the regular polygon, must be different from 180°.

Since PM and QN have but one point of intersection, there is one and only one circle through A, B, and C.

We must now show that D lies on this circle through A, B, and C. Draw OA, OB, OC, and OD (Fig. 36).

\[
\begin{align*}
\text{In triangles } OBA \text{ and } OCD, \quad & OB = OC \quad \text{(Why?)} \\
\text{and } \quad & BA = CD. \quad \text{(Why?)} \\
\text{Moreover, } \quad & \angle CBA = \angle BCD \quad \text{(Why?)} \\
\text{and } \quad & \angle COB = \angle BCO. \quad \text{(Why?)} \\
\text{Therefore } \quad & \angle OBA = \angle OCD.
\end{align*}
\]

Therefore OA = OD (Why?) and D lies on the circle through A, B, and C.

In similar manner we can now prove that E, F, . . . must lie on the circle through A, B, C, and D.
Theorem 24. A circle can be inscribed in any regular polygon.

GIVEN: Regular polygon $ABCD$ . . . . See Fig. 37.

TO PROVE: That a circle can be inscribed in $ABCD$ . . . .

PROOF: Let 0 be the center of the circle circumscribed about polygon $ABCD$ . . . . The sides of the polygon are equal chords of this circumscribed circle and are therefore equally distant from the center O. See Ex. 4, page 137, or use Principles 10 and 12 directly.

Consequently a circle having 0 as center and radius equal to the perpendicular distance $r$ from 0 to a side of the polygon will be tangent to each side of the polygon (by Theorem 20). This circle is therefore inscribed in the polygon.

The common center 0 of the inscribed and circumscribed circles is often called the CENTER OF THE POLYGON.

EXERCISES

*1. Prove that through three points not on a straight line there is one and only one circle. That is, prove that a circle can be circumscribed about any triangle.

2. Prove that the line joining the center of a regular polygon to a vertex of the polygon bisects the angle at the vertex.

*3. Prove that an equilateral polygon inscribed in a circle is a regular polygon.

4. Draw two examples of inscribed equiangular polygons that are not regular.
5. Prove that an equiangular polygon circumscribed about a circle is a regular polygon.

6. Draw two examples of circumscribed equilateral polygons that are not regular.

7. Prove that if a circle is divided into any number of equal arcs, the chords of these arcs form an inscribed regular polygon.

8. Prove that if a circle is divided into any number of equal arcs, the tangents at the points of division form a circumscribed regular polygon.

9. Prove that lines drawn from each vertex of an inscribed regular polygon to the mid-points of the adjacent arcs form an inscribed regular polygon of double the number of sides.

10. Prove that tangents at the mid-points of arcs between adjacent points of contact of the sides of a circumscribed regular polygon form with the sides of the given polygon a circumscribed regular polygon of double the number of sides.

11. Prove that two regular polygons of the same number of sides are similar. Suggestion: Show angles equal and corresponding sides proportional.

12. Given a square of side 5 inches, compute the radius of the circumscribed circle.

13. Given a circle of radius 7 inches, compute the side of the inscribed square.

14. Given an equilateral triangle of side 4 inches, compute the radius of the circumscribed circle.

15. Given a circle of radius 7 inches, compute the side of the inscribed equilateral triangle.

16. Given a circle of radius 5, compute the perimeter of the circumscribed equilateral triangle.
SUMMARY

DEFINITIONS: circle, center, radius, equation of circle, arc, central angle, minor arc, major arc, semicircle, equal arcs, mid-point of arc, equal circles, chord, diameter

19. In the same circle, or in equal circles, equal chords have equal arcs; and conversely.

DEFINITIONS: secant, intersection of line and circle, tangent, point of tangency, point of contact, inscribed and circumscribed polygons

20. A line perpendicular to a radius at its outer extremity is tangent to the circle.

21. Every tangent to a circle is perpendicular to the radius drawn to the point of contact.

21a. There is only one tangent to a circle at any given point of the circle.

DEFINITIONS: line of centers, intersection of two circles, common chord, tangent circles, common internal tangent, common external tangent, inscribed angle

22. An inscribed angle is equal to half the central angle having the same arc.

22a. Equal angles inscribed in the same circle have equal arcs.

22b. All inscribed angles having the same arc are equal.

22c. Every angle "inscribed in a semicircle" is a right angle.

23. A circle can be circumscribed about any regular polygon.

24. A circle can be inscribed in any regular polygon.
EXERCISES

1. Two circles are tangent externally at $T$ and touch a common external tangent at $R$ and $S$. Prove that angle $RTS$ is a right angle.

2. Consider what happens in Ex. 4, page 147, as the two circles approach the position of internal tangency or withdraw toward the position of external tangency. Prove that if two circles are tangent, either internally or externally, and if two lines drawn through the point of contact are terminated by the circles, the chords joining the ends of these lines are parallel. Prove this for both cases, when the circles are tangent internally and when they are tangent externally (Fig. 38).

In proving any theorem involving tangent circles it is usually helpful to draw the common tangent.

3. Two circles are tangent at $T$. Through $T$ three lines are drawn, cutting one circle in $A$, $B$, $C$ and the other in $A'$, $B'$, $C'$ respectively. Prove that the triangles $ABC$ and $A'B'C'$ are similar. Prove this if the circles are tangent internally and if they are tangent externally.

4. Prove that if two circles are tangent internally, all chords of the greater circle that are drawn from the point of contact are divided proportionally by the smaller circle.

5. Two circles are tangent at $T$. From the ends $A$, $B$ of a diameter of one circle lines $AT$, $BT$ are drawn cutting
the other circle at $A', B'$ respectively. Prove that $A'B'$ is a diameter of the second circle. Prove this for both cases, when the circles are tangent internally and when they are tangent externally.

6. What interesting properties have the points $P$ and $Q$ and the line segment $PQ$ in Fig. 39? Prove that you are right.

7. Prove that the tangents to two intersecting circles from any point on the extension of the common chord are equal.

8. Prove that the radius of the circle inscribed in an equilateral triangle is equal to one-half the radius of the circumscribed circle and to one-third the altitude of the triangle.

9. In Fig. 40 two circles intersect at the points $A$ and $B$. Through $A$ a secant is drawn at random, cutting the circles at $C$ and $D$. Prove that however the secant is drawn, the angle $DBC$ is constant.

10. Three circles, $O$, $P$, and $Q$ (Fig. 41), are tangent externally at points $A$, $B$, and $C$, and chords $AB$ and $AC$ are extended so as to cut circle $Q$ at $D$ and $E$. Prove that $DE$ passes through the center of circle $Q$. Suggestion: Show $LOAC = LCEO$ and $LPAB = LBDQ$. 

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11. When two perpendicular radii are extended outside a circle, as in Fig. 42, they are cut at A and B by a tangent to the circle. Prove that the other tangents from A and B are parallel.

12. Prove that if two common external tangents or two common internal tangents are drawn to two circles, the segments of the tangents between the points of contact are equal.

13. Prove that every equiangular polygon inscribed in a circle is regular if the polygon has an odd number of sides. 
*Suggestion:* Try first to prove this for an equiangular polygon of 5 sides. Will your proof apply to a polygon of 6 sides? Of 7 sides?

14. Prove that every equilateral polygon circumscribed about a circle is regular if the polygon has an odd number of sides.

15. Prove that if three circles intersect one another, their common chords are concurrent. 
*See Fig. 43. Suggestion:* Two of these common chords, say AB and CD, must meet at some point O. In the special case that this point O is an end-point of each of these two chords, it is the common intersection of all three circles. The proof of the theorem for this special case is obvious.

If now 0 has not this special position, it will not be on any of the given circles. Join E to 0 and assume that EO extended meets the circles that intersect at E in two different points, P and Q. Prove that OP = OQ (see Ex. 35, page 151), whence P and Q coincide.
16. A plane cuts a cone at right angles to its axis. What sort of curve is the intersection of the plane and cone?

17. Describe the intersection of a plane and cylinder when the plane is perpendicular to the axis of the cylinder. If the plane cuts the axis at an oblique angle, the intersection is an ellipse. Describe the position in which you must hold a fifty-cent piece under an overhead light in order to cast a circular shadow on the floor; on the tilted cover of a book. What positions of the coin will give an elliptic shadow on the floor? On the tilted cover of a book?

18. What sort of curve is formed by the intersection of a plane and a sphere? Assuming the earth to be a perfect sphere, compare the size of any meridian and the equator. The planes that cut the sphere at a meridian or at the equator pass through the center of the sphere. Such sections of the sphere are called "great circles." What do you know about the centers of all great circles of a sphere? What do you observe about the size of the circles that form the various parallels of latitude on the surface of the sphere? What can you say about the centers of all these circles? The sections of a sphere formed by planes that do not pass through the center are called "small circles."

19. Show that through any two given points on the surface of a sphere an arc of a great circle can be drawn. How must the two points be chosen if they can be connected by the arcs of more than one great circle?

20. The shortest line that can be drawn on the surface of a sphere, connecting two given points of the sphere, is an arc of a great circle. Draw on a globe the great circle air route between New York and Paris. Where does it leave North America? Where does it first meet Europe?
21. The angle between two meridians is often thought of as measured in degrees along the equator. The meridian through Philadelphia meets the equator 75° west of the meridian that passes through Greenwich, England. What is the difference in longitude between Philadelphia and Greenwich? What is the difference in longitude between these same two meridians at the Arctic Circle?

22. What is the difference in time between Greenwich and Philadelphia clocks?

23. The sides of a spherical polygon are arcs of great circles of a sphere. Show that Ex. 11, page 155, is not true of spherical polygons. Consider the case of equilateral spherical triangles.

**EXERCISES**

You have gone far enough now in geometry to understand the nature of a proof and to appreciate the importance of assumptions, definitions, and undefined terms in a logical science. The following exercises require you to consider underlying assumptions and to examine critically the reasoning that is employed. Point out all the incorrect or doubtful statements that you can find.

1. Triangles \(JKL\) and \(PQR\) are equal because in triangle \(JKL\) two sides and an angle opposite one of them are equal respectively to two sides and an angle opposite one of them in triangle \(PQR\).

2. In the trapezoid \(ABeD\) in Fig. 44 the line joining the mid-points \(E\) and \(F\) of the non-parallel sides is parallel to the bases. This is shown as follows. Through \(E\) draw a parallel to \(AB\) meeting \(BD\) at \(M\); and similarly draw \(FM\)

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parallel to $CD$. $FM$ will be parallel to $AB$ as well. Consequently $EMF$ is a straight line parallel to both bases of the trapezoid.

3. Definition: Two lines are said to be parallel if they are everywhere equidistant.

4. "'England expects every man to do his duty' is a battle-cry that has gone down in history. It should govern the mimic battles of the athletic field as well. We would make no unreasonable demands. But we expect every team that represents this school to go down in history as having won the majority of its games!" (Applause)

5. In the inquiry following the collapse in mid-air of the Navy dirigible "Shenandoah" in an Ohio thundersquall, Mrs. Lansdowne testified as follows: "I told him [Captain Foley] I would emphasize a fact which the court had evaded—that the 'Shenandoah' had been sent on a political flight, despite Secretary [of the Navy] Wilbur's denial. Captain Foley said I had no right to say it was a political flight, as the taxpayers had a right to see their property."

6. A salesman from the Seacoal Coke Company was urging me to buy coke from his company at $12 a ton rather than from the local gas works at $11 a ton. "For," he argued, "while our coke costs more, at the same time it yields 900 more heat units per ton than does the gas works' coke."

7. Public-school graduates do better work in college than graduates of private schools. Therefore the public schools afford the better college preparation.

8. "Alumni are usually unanimous in expressing their belief that the pursuit of outside activities is the best training for a business career. And it is for such careers
that our colleges must now train their graduates, no mat-
ther what their fundamental purpose was one hundred
years ago. It is worth noting in passing that no less
important a figure in the business world than Blank,
who is president of one of the largest business enterprises
in the country, made a statement only last week at a
reunion on the campus that his greatest mistake while
here in school was his failure to get into activities.

Determine the assumptions underlying each of the fol-
lowing statements.

9. Advertisement: At last! The car with self-emp-
tying ash trays! The 1940 Sheraton!

10. Advertisement: 8 men out of 10 will choose gray
for their new fall hats. That's why grays predominate
in our stock.

11. You should invest in this Get-Rich-Quick Com-
pany; in so doing you take only the usual risk inherent in
any new business venture.

12. In 1924 our nation was urged to vote the Republi-
can ticket on the ground that never in our history had
we been so prosperous as during the Republican adminis-
tration from 1921 to 1924.

13. A member of the city council made a stirring plea
for the cancellation of all tax bills remaining unpaid for
more than five years. He based his appeal on the state-
ment that poor widows had all they could do to make
ends meet and that the city government should wish to
appear as protector rather than as oppressor of the un-
fortunate.
This photograph shows draftsmen at work in the drafting room of a large airplane factory on the Pacific coast. These men make frequent use of many of the geometric constructions set forth in the next chapter. Your task in this chapter will be to prove the correctness of these constructions.
WE CAN MAKE all the constructions necessary to this geometry, or to any other elementary geometry, by means of three instruments: a ruler marked in inches, centimeters, or the like; a protractor; and a pair of compasses. The marked ruler, often called a scale, and the protractor embody our fundamental assumptions about numbering distances and angles as set down in Principles 1 and 3. We need the compasses in order to draw circles. Architects and draftsmen, who are continually making geometric constructions, find these instruments indispensable.

From the time of the ancient Greeks down to the present, geometers have been fascinated by the question of what constructions are possible with only two very simple instruments: the straightedge, or unmarked ruler, and compasses. This question is interesting also because of its intimate relation to algebra. Consider, for example, any geometric construction in which we attempt to construct a desired length, \( x \), from its relation to given lengths, \( a, b, c \), using only a straightedge and compasses. Whether or not we can do so depends upon the algebraic relationship of \( x \) to \( a, b, c \). We can construct the desired length \( x \) with these two instruments whenever the relationship of \( x \) to
a, b, c. ••• involves ultimately only addition, subtraction, multiplication, division, and the extraction of square roots. In all other cases it is impossible to construct \( x \) with the use of only a straightedge and compasses. Thus with these two instruments we can construct a length \( x \) equal to \( 3a + 2b - \sqrt{ab} \), when \( a \) and \( b \) are known, but we cannot construct a length \( x \) equal to \( \sqrt{abc} \). In general, it is impossible to divide a given angle into a given number of equal parts by means of straightedge and compasses alone. This constitutes the chief difference between constructions that are possible with these two instruments and constructions that are possible with scale, protractor, and compasses.

First we shall consider constructions that are done by means of the three instruments, scale, protractor, and compasses. Then we shall see how many of these are possible when we use only the straightedge and compasses. Probably you are already familiar with some of these constructions. Now, however, you can prove the correctness of such constructions.

By means of the marked ruler or scale we can draw a line through two points, measure a line segment, layoff a line of given length, and divide a line segment into \( n \) equal parts. Thus, if the line segment \( AB \) is to be divided into six equal parts, we can use any scale we like to measure its length, then take one-sixth of this length and lay it off five times from \( A \) or \( B \). Fig. 1 shows two ways of doing this.

\[
\begin{array}{cccccc}
0 & 10 & 16 & 19 & 22 & 25 \\
6 & 13 & 16 & 19 & 22 & 25 \\
A & 10 & 13 & 16 & 19 & B \\
\end{array}
\]

Fig. 1

The numbers corresponding to \( A \) and \( B \) may be given to us, instead of being found by means of a scale. Thus in Fig. 2 on the next page we are given the straight line
with the point $A$ corresponding to $3\frac{1}{2}$ and the point $B$ corresponding to $7\sqrt{2}$. If we have to divide this line segment $AB$ into six equal parts, we must find the numbers corresponding to the five points of division. If these numbers are to be found correct to four figures, we must express $3\frac{1}{2}$ and $7\sqrt{2}$ with four-figure accuracy before we subtract and divide by 6.

\[
\begin{array}{cccccc}
3\frac{1}{2} & 4.269 & 5.395 & 6.521 & 7.647 & 8.773 \\
A & ! & ! & ! & ! & ! & B \\
\end{array}
\]

Fig. 2

Similarly, if $AB$ is to be divided into three parts proportional to the given lengths $l$, $m$, and $n$, we measure the given lengths with the scale and layoff lengths equal to the following fractional parts of $AB$:

\[
\frac{l}{l+m+n'} \quad \frac{m}{l+m+n'} \quad \frac{n}{l+m+n'}
\]

On a sheet of paper layoff lines equal in length to $AB$, $l$, $m$, and $n$, as given in Fig. 3. Then divide $AB$ into three parts proportional to $l$, $m$, and $n$.

\[
\begin{array}{ccc}
\text{A} & \text{m} & \text{B} \\
\text{n} & &
\end{array}
\]

Fig. 3

Another construction frequently needed in geometry is to layoff the length that is the fourth proportional to three given lengths. That is, given the three lengths $q$, $r$, and $s$ in Fig. 4, we must layoff the length $t$ such that $q = \frac{s}{r}$ after we have meas-
ured the lengths \( q, r, \) and \( s \), we can calculate the value of \( t \) from the equation \( t = \frac{rs}{q} \) and then layoff this length on any desired line.

Find the length of the fourth proportional to \( q, r, \) and \( s \) in Fig. 4 on page 167.

The principles underlying the constructions which have just been discussed can also be applied to angles by means of the protractor. For example, we can divide a given angle into seven equal parts, divide it also into \( n \) parts proportional to \( n \) given angles, and find the fourth proportional to three given angles.

PERPENDICULARS. To draw the perpendicular to a given line at a given point of the line, we need only the protractor. When the given point \( P \) is not on the given line \( AB \), however, we may use the construction employed in proving Principle 11. See Fig. 5.

To draw a line from \( P \) perpendicular to \( AB \), first join \( P \) to any point \( Q \) of \( AB \). Construct \( QR \) so that \( LBQR \) equals \( LBQP \). On \( QR \) layoff \( QP' \) equal to \( QP \). Then \( PP' \) will be perpendicular to \( AB \).

PARALLELs. To draw the line through \( P \) parallel to \( AB \), first draw the perpendicular through \( P \) to \( AB \) by the previous construction, and then by means of a protractor draw the perpendicular to this perpendicular at \( P \). See Fig. 6.
EXERCISES

In making the constructions called for in the following exercises, do not mark any lines or other figures printed in the book. First copy the given figure and then make the construction on your own copy of the figure. Use only scale and protractor in these exercises. In each case you will have to decide whether to use an inch-scale or a centimeter-scale.

1. Divide the line segment $CD$ in Fig. 7 into five equal parts. Use as finely divided a scale as you can obtain to help you mark off the required points as accurately as possible.

2. Divide line segment $EF$ (Fig. 8) into six equal parts.

3. What numbers should be assigned to the points which divide $GH$ in Fig. 9 into four equal parts?

4. Divide the line segment $CD$ in Fig. 7 into two parts proportional to the two line segments $l$ and $m$ in Fig. 10.

5. Divide $EF$ in Fig. 8 into three parts proportional to the line segments $r$, $s$, and $t$ in Fig. 11.
6. Find the fourth proportional to the three lengths \( r, s, \) and \( t \) in Fig. 11 on page 169.

7. Find the fourth proportional to the three lengths \( t, s, \) and \( r \) in Fig. 11.

8. Divide angle \( AOB \) (Fig. 12) into three equal parts.

9. Divide the angle \( AOB \) into parts proportional to the angles \( COD \) and \( EOF \) shown in Fig. 13.

10. Find the fourth proportional to the angles \( AOB, COD, \) and \( EOF. \)

11. In Fig. 14 draw the line through \( P \) perpendicular to \( AB. \)

12. In Fig. 14 draw the line through \( P \) parallel to \( AB. \)

CIRCLES. So far we have considered only constructions that require scale and protractor. Certain other constructions that demand the actual drawing of a circle require compasses in addition to these two instruments. We shall not actually make such constructions at this time, however, because the most important of them will be done later by straightedge and compasses alone; it will be evident then how we ought to proceed in case we wish to use scale, protractor, and compasses instead.

REGULAR POLYGONS. If you had to construct with scale and protractor a regular polygon of 7, or of \( n, \) sides such that each side would be equal to a given line segment 170
AB, you would need to recall what you have learned in
Exercises 6 and 10, page 85, about the angles of a re­
gular polygon. But once you had discovered how large each
angle of the polygon would have to be, you would have
no trouble in completing the construction.

Also, if you had to inscribe a regular polygon of n sides
in a given circle, using only scale and protractor, you
would need only to consider the relation between the
sides of the polygon and angles at the center of the circle
in order to figure out how to make the construction.

It is clear that any construction involving the laying
off of lengths and angles can be made with scale and
protractor, and to any desired degree of accuracy. Thus
we can make the above two constructions concerning
regular polygons as accurately as we wish by means of
scale and protractor. But we cannot make these two
constructions with straightedge and compasses except in
certain special cases. For each of these constructions in­
volves the division of 180° or 360° into n equal parts, and
ordinarily we cannot divide an angle into n equal parts
with straightedge and compasses. As stated previously,
this constitutes the chief difference between constructions
that are possible with scale, protractor, and compasses
and those that are possible with only straightedge and
compasses.

EXERCISES

1. Using scale and protractor, construct a regular octagon with
each of its sides equal to AB in Fig. 15.

2. Construct a regular polygon of nine sides with
each of its sides equal to AB in Fig. 15, using scale and
protractor.
3. Inscribe a regular octagon in a given circle, using scale and protractor.

4. Inscribe a regular polygon of nine sides in a given circle.

From a practical standpoint, constructions with straightedge and compasses are neither more accurate nor less accurate than constructions with scale and protractor. Moreover, as stated above, there are some scale and protractor constructions that cannot be done by straightedge and compasses. Theoretically, however, those constructions that can be done by straightedge and compasses are absolutely accurate.

From this point on in this chapter the constructions that we shall consider will be done by means of straightedge and compasses.

*To construct the perpendicular bisector of a line segment.*

**Fig. 16**

**Fig. 17**

**GIVEN:** Line segment \( AB \) (Fig. 16).

**TO CONSTRUCT:** The perpendicular to \( AB \) at its midpoint.

**ANALYSIS:** Let us assume that the construction has been performed, that \( PQ \) is the required perpendicular, and
see what we can learn from the completed figure. All the points on \( PQ \) are equidistant from \( A \) and \( B \) (by Principle 10). We have, therefore, to establish two points that are equidistant from \( A \) and \( B \) in order to determine the line \( PQ \).

**Construction:** With \( A \) as center and a radius greater than \( \frac{1}{2}AB \) draw an arc. See Fig. 17 on page 172. With \( B \) as center and the same radius draw a second arc.

The arcs intersect at the points \( R \) and \( R' \). (See Ex. 13, page 142.) The line through \( R \) and \( R' \) will be perpendicular to \( AB \) at its midpoint. Prove it.

Is it possible to choose too small a radius for the arcs? Too large a radius?

In any construction involving the intersection of arcs you ought to try to have the arcs cross at approximately right angles. If you draw arcs that cross like those in Fig. 18, you cannot be sure of the exact location of the crossing. This is purely a practical matter. Theoretically, one sort of intersection is as good as another.

*To construct an angle equal to a given angle.*

**Given:** Angle \( A \) (Fig. 19).

**To Construct:** An angle equal to \( \angle A \) with vertex at \( A' \).

Complete the construction. Prove that your construction is correct.
To bisect a given angle.

**GIVEN:** Angle \( \angle A \) (Fig. 20).

**TO CONSTRUCT:** An angle equal to \( \frac{1}{2} \angle A \).

Complete the construction and prove that it is correct.

Through a given point, to draw the line that is parallel to a given line.

**GIVEN:** Line \( l \) and point \( P \) not on \( l \).

**TO CONSTRUCT:** The line through \( P \) parallel to \( l \).

Three methods of making the construction are discussed below.

**First method:** Through \( P \) draw a random line \( t \) cutting \( l \) at \( A \) (Fig. 21). Then construct an angle with vertex at \( P \) and with one side lying along \( t \) so that the angle is equal to, and corresponds to, one of the angles with vertex at \( A \). The other side of this angle establishes the parallel to \( l \) through \( P \). Why?

**Second method:** With any convenient point \( O \) as center (Fig. 22) draw a circle through \( P \) intersecting \( l \) at \( A \) and \( B \). With \( B \) as center and \( AP \) as radius draw an arc intersecting the circle at \( Q \). \( PQ \) will be parallel to \( AB \). Why? Suggestion: Show by construct-
To divide a given line segment into parts proportional to \( n \) given line segments.

GIVEN: Line segments \( AB, p, q, r, \) and \( s \) (Fig. 24).

TO CONSTRUCT: Points on \( AB \) which divide it into segments proportional to \( p, q, r, \) and \( s \).

ANALYSIS: Let us assume that the construction has been made and that \( H, J, K \) are the desired points. If now the segments \( p, q, r, \) and \( s \) are laid off in order along the line \( AX \) (drawn at random through \( A \)), then \( HP, JQ, KR, \) and \( BB \) will all be parallel. Why?

CONSTRUCTION: If therefore we layoff the given segments \( p, q, r, \) and \( s \) in order along a line \( AX \) and draw \( BB \), it will be easy to construct the points \( H, J, \) and \( K \). How?

Complete the construction and prove that your construction is correct.
EXERCISES

1. How could the construction discussed on page 176 be used to divide a given line segment into \( n \) equal parts? Divide a line 2 inches long into 5 equal parts by this method.

2. Another construction for dividing a given line segment into \( n \) equal parts is shown in Fig. 25.

CONSTRUCTION: With any convenient radius and with \( A \) and \( B \) as centers draw two arcs intersecting in points \( P \) and \( Q \). Draw \( AP \) and extend it to a point \( R \) such that \( AR \) is equal to \( (n-1)AP \). Draw \( RQ \). It intersects \( AB \) at \( N \), making \( BN \) equal to \( \frac{1}{n} \) of \( AB \). Hence \( BN \) is one of the required equal parts of \( AB \). Why? Suggestion: Draw \( BQ \) and use triangles \( BNQ \) and \( ANR \).
To construct the fourth proportional to three given line segments.

\[
\begin{array}{c}
\text{k} \\
A \quad k \quad \ell \quad m \\
\end{array}
\]

GIVEN: Three line segments \( k \), \( l \), and \( m \) (Fig. 26).

TO CONSTRUCT: Line segment \( n \) such that \( \frac{k}{l} = \frac{m}{n} \).

ANALYSIS: Let us assume that \( n \) has been constructed. If then \( k \) and \( m \) are laid off in order along the line \( AX \), and \( l \) and \( n \) are laid off in order along \( AY \), \( MN \) will be parallel to \( KL \). Why?

CONSTRUCTION: Draw any two intersecting lines \( AX \) and \( AY \). On \( AX \) layoff lengths \( k \) and \( m \), and on \( AY \) layoff length \( l \).

Complete the construction and prove that it is correct.

To construct the perpendicular to a given line at a given point of the line.

GIVEN: Line \( l \) and point \( P \) on \( l \).

TO CONSTRUCT: The perpendicular to \( l \) at \( P \).

CONSTRUCTION: Fig. 27 suggests the method to use. Make the construction and prove that it is correct.
The constructions shown in Figs. 28, 29, and 30 are sometimes used in constructing a perpendicular to $l$ at a point $P$ very near the edge of the paper. (In Fig. 30 point $O$ is taken at random, and a circle with radius $OP$ is drawn.) Such constructions have little practical value, but it is interesting to see why each of them is correct. We shall do this in Ex. 5 on page 181.

To construct through a given point the perpendicular to a given line.

**GIVEN:** Line $l$ and point $P$.

**TO CONSTRUCT:** The perpendicular to $l$ through $P$.

**CONSTRUCTION:** Fig. 31 suggests the method to use. Complete the construction and prove that it is correct.
Which is easier, to make this construction with straight-edge and compasses or with scale and protractor?

Show that when $P$ lies on $l$ this construction can still be performed, and that it is equivalent to the construction shown in Fig. 27 on page 179.

To construct the mean proportional between two given line segments.

**Given:** Line segments $a$ and $b$ (Fig. 32).

**To construct:** A line segment $m$ such that $\frac{a}{m} = \frac{m}{b}$.

![Fig. 32]

**Analysis:** The construction of the fourth proportional to three given line segments gives us no hint here. We have met a mean proportional before in Ex. 14, page 85, and in Ex. 26: page 150, however, and may get a suggestion from these sources.

Complete the construction and prove that it is correct.

The length of the required mean proportional $m$ satisfies the equation $m^2 = ab$ or $m = \sqrt{ab}$. In particular, if $b = 1$, $m = \sqrt{a}$. We may use this method, therefore, to construct a line segment of length equal to $\sqrt{5}$, or the like, provided we know also the unit length.

**Exercises**

1. Given an arc of a circle, find its mid-point.

2. Given the rectangle $ABCD$ and point $E$ on $AB$ (Fig. 33), construct a rectangle similar to $ABCD$ and with one side equal to $AE$.  

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3. Given the unit length, construct line segments of lengths $\sqrt{7}$ and $\sqrt{3.5}$.

4. Construct the fourth proportional to three line segments of lengths 5, $2\sqrt{3}$, and $3+\sqrt{2}$.

5. Prove the correctness of the constructions shown in Figs. 28-30, page 179.

To construct the circle through three given points.

GIVEN: Points $A$, $B$, $C$, not on the same straight line. See Fig. 34.

TO CONSTRUCT: The circle passing through $A$, $B$, and $C$.

ANALYSIS: Since the center of the required circle must be equidistant from $A$, $B$, and $C$, we have to locate a point satisfying this condition. We have seen in Theorem 10 that all points equidistant from the ends of a line segment lie on the perpendicular bisector of the line segment. Therefore the center of the circle through $A$, $B$, and $C$ must lie on the perpendicular bisectors of $AB$, $BC$, and $AC$. Show that these three perpendicular bisectors meet in a point. This can best be done by showing first that two of these meet in a point, and then that their point of intersection lies on the third.

CONSTRUCTION: Fig. 34 suggests the method to use. The work involved in constructing the perpendicular bisectors can be shortened in this case by keeping the same radius throughout, as shown in Fig. 34. Make the construction and prove that it is correct.

*That there is such a circle has been proved in Ex. 1 on page 154.
To circumscribe a circle about a given triangle.

This construction is equivalent to the preceding one.

Given an arc of a circle, to find its center.

**GIVEN:** An arc of a circle (Fig. 35).

**TO CONSTRUCT:** The center of the circle.

**ANALYSIS:** The center, when found, will be equidistant from all points of the arc, in particular from the three points \( A, B, \) and \( C \) chosen at random. Where are all the points that are equidistant from \( A \) and \( B \)? From \( B \) and \( C \)?

Make the construction and prove that it is correct.

To inscribe a circle in a given triangle.

**GIVEN:** Triangle \( ABC \) (Fig. 36).

**TO CONSTRUCT:** A circle tangent to the three sides of triangle \( ABC. \) *

**ANALYSIS:** Assume the construction performed. Then the center \( 0 \) must be equidistant from \( AB, BC, \) and \( CA. \) From this it follows that in the right triangles \( ADO \) and \( AFO, LDAO = LFAO. \) Why?

---

*That there is such a circle can be proved by showing (1) that the bisectors of any two angles of a triangle must intersect, and (2) that this point of intersection is equally distant from the three sides of the triangle and so can serve as center of the inscribed circle. We shall return to this idea in Ex. 19 on page 256.*
That is, 0 lies on the bisector of angle \( BAC \). Similarly O lies on the bisectors of angles \( B \) and C.

**Construction:** Draw the bisectors of the three angles, \( A, B, \) and \( C \). They will meet at the center of the inscribed circle.

How would you construct a circle tangent to one side of a triangle and also tangent to the other two sides extended, as in Fig. 37?

**At a given point of a given circle to construct the tangent to the circle.**

**Given:** Circle \( O \) and point \( T \) on circle \( O \) (Fig. 38).

**To Construct:** The tangent to circle \( O \) at \( T \).

**Analysis:** When constructed, the tangent at \( T \) will be perpendicular to the radius \( OT \).

Make the construction and prove that it is correct.

**Through a given point outside a given circle to construct a tangent to the circle.**

**Given:** Circle \( O \) and point \( P \) outside circle \( O \).

**To Construct:** A tangent through \( P \) to circle \( O \).

**Analysis:** Let us assume that the construction has been performed (Fig. 39). There will be two tangents, \( PA \) and \( PE \). What do you know about the size of angles \( PAO \) and...
To find $A$, we must be able to construct a right triangle $POA$ having $PO$ as hypotenuse and having one side equal to the radius of the circle.

**CONSTRUCTION:** Find the mid-point $M$ of $OP$ (Fig. 40) and on $OP$ as diameter draw a circle cutting the given circle at $A$ and $B$. Draw $PA$ and $PB$.

Prove that these two lines will be tangent to circle $O$.

*To construct the common tangents to two given circles.*

**GIVEN:** Circles $O$ and $O'$ with radii $r$ and $r'$ respectively. See Fig. 41.
TO CONSTRUCT: Common internal and external tangents. See Exercises 17 and 21, pages 144 and 145.

CONSTRUCTION: With 0 as center construct a circle with radius equal to $r - r'$. Tangents from 0' to this circle will be parallel to the common external tangents. Why? What if $r$ and $r'$ are equal?

How would you construct the common internal tangents? See the lower diagram in Fig. 41 on page 184.

Construct the common tangents and give a complete proof to show that your construction is correct.

EXERCISES

Draw three pairs of circles like those shown in Fig. 42. For each pair you have drawn construct all the possible common tangents.

Fig. 42

To construct a triangle when the three sides are given.

GIVEN: Lengths $l$, $m$, and $n$ (Fig. 43).

TO CONSTRUCT: A triangle having lengths $l$, $m$, and $n$ for sides.

Construct the triangle.

Can the lengths be so chosen as to render the construction impossible? How?
To construct a triangle when two sides and the included angle are given.

GIVEN: Lengths $l$ and $m$ and angle $A$ (Fig. 44).

TO CONSTRUCT: A triangle having lengths $l$ and $m$ for two of its sides and having angle $A$ included between them.

Construct the triangle.

To construct a triangle when two angles and a side are given.

GIVEN: Angles $A$ and $B$ and length $l$ (Fig. 45).

TO CONSTRUCT: A triangle having two of its angles equal respectively to $A$ and $B$ and having one side equal to $l$.

CONSTRUCTION: Case 1. Angles $A$ and $B$ have side $l$ in common.

Case 2. Angles $A$ and $B$ do not have side $l$ in common. In this case the third angle can be found.

Construct the triangle and supply the proof.

To construct a triangle when two sides and the angle opposite one of them are given.

GIVEN: Angle $A$ and sides $l$ and $m$. See Fig. 46 on the next page. Notice that the given angle $A$ may be an acute, an obtuse, or a right angle, and that $m$ may be given less than, equal to, or greater than $l$.

TO CONSTRUCT: A triangle having two of its sides equal to $l$ and $m$ and having angle $A$ opposite side $m$. 

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CONSTRUCTION: Construct an angle equal to angle \( A \) and layoff on one side of the angle a length \( AB \) equal to \( l \). With \( B \) as center and \( m \) as radius draw an arc.

Try to discover all possible cases, according as \( m \) is less than, equal to, or greater than \( l \); and according as \( A \) is acute, obtuse, or a right angle. Notice that this construction is impossible for a given angle \( A \) when \( m \) is much shorter than \( l \). In some cases two triangles can be found to satisfy the given conditions; in other cases, only one; and in some cases, none at all. In two cases two trian-
gles seem at first to be possible; but closer examination shows that one triangle contains, not angle \( A \), but an angle equal to \( 180° - A \). Which are these two cases? To discover all possible cases you must adopt some systematic procedure. The diagrams in Fig. 46 suggest one way of doing this. You may prefer some other way.

**EXERCISES**

Plato said, "The triangle which we maintain to be the most beautiful of all the many triangles is that of which the double forms a second triangle which is equilateral." If we judge the beauty of a triangle by its power to furnish other interesting geometrical figures by combination, no other triangle compares with this favorite triangle of Plato. Out of it can be built the figures shown in Fig. 47 and three of the five regular solids. See Figs. 48-50.

1. Construct a \( 30°_60° \) right triangle and construct from that the diamond-shaped figure shown in Fig. 47.

2. A regular tetrahedron is a solid each of whose four faces is an equilateral triangle. Construct one out of thin cardboard, with appropriate flaps for pasting. See Fig. 48. Before folding along an edge score lightly with a knife on the outside of the fold.
3. The regular octahedron has eight faces, each an equilateral triangle. Construct one out of thin cardboard, with appropriate flaps for pasting. See Fig. 49. Make each edge two inches long.

![Fig. 49](image1)

![Fig. 50](image2)

4. The regular icosahedron has twenty faces, each an equilateral triangle. Construct one out of thin cardboard, with appropriate flaps for pasting. See Fig. 50. Make each edge $1\frac{1}{2}$ inches long.

The first two of these regular solids, together with the cube, were known to the ancient Egyptians; the Greeks discovered the regular icosahedron and dodecahedron (see Ex. 15, page 196). These five are the only regular solids that there can be. See pages 219 and 220.

*To trisect a right angle.*

**GIVEN:** Right angle $AOB$ (Fig. 51).

**TO CONSTRUCT:** Two half-lines from $O$ which divide $\angle AOB$ into three equal parts.

**CONSTRUCTION:** With $O$ as center and any convenient radius draw an arc intersecting $OA$ at $C$ and $OB$ at $D$. With the same radius and with $C$ as center draw an arc intersecting the first arc at $F$. Locate $E$ similarly, using $D$ as center. $OE$ and $OF$ are the required half-lines.

Prove that the construction is correct.
The right angle is one of the relatively few angles which can be trisected by straightedge and compasses, whereas any angle can be trisected by means of the protractor. Here again we observe this important and significant difference between the constructions which are possible by straightedge and compasses and those which can be performed by scale and protractor.

*To inscribe regular polygons of six, three, and twelve sides in a given circle.*

---

**GIVEN:** Circle with center O.

**TO CONSTRUCT:** A regular hexagon with its vertices on the circle.

**CONSTRUCTION:** With any point $A$ of the circle as center, and with radius $OA$, draw an arc cutting the circle at $B$ and $F$. See Fig. 52. Do the same at $B$ and $F$ and continue until points $C$, $D$, and $E$ have been located. These six points will be the vertices of the inscribed regular hexagon.

Supply the proof.

How would you inscribe a regular triangle in a given circle? How would you inscribe a regular dodecagon?

*To inscribe regular polygons of four, eight, and sixteen sides in a given circle.*

**GIVEN:** Circle with center O.
TO CONSTRUCT: A square with its vertices on the circle.

CONSTRUCTION: Draw any diameter $AB$ and construct its perpendicular bisector, $CD$. See Fig. 53. $ABCD$ is the required square.

How would you inscribe a regular octagon in a given circle? How would you inscribe a regular polygon of 16 sides?

To inscribe regular polygons of five and ten sides in a given circle.

GIVEN: Circle with center $O$ and radius $r$.

TO CONSTRUCT: A regular pentagon with its vertices on the circle; also a regular decagon.

CONSTRUCTION: Draw any diameter $AB$ and construct its perpendicular bisector, $CD$. See the first diagram in Fig. 54. Find the mid-point, $M$, of $OB$ and with $M$ as center and radius $MC$ draw an arc intersecting $AO$ at $E$. $CE$ is the length of a side of the inscribed regular pentagon. With $C$ as center and radius $CE$ we can locate the
vertex $P$ of the pentagon. The other three vertices, $Q$, $R$, and $S$, are now easily found.

We can inscribe a regular decagon in the same circle by finding the mid-points of the arcs $CP$, $PQ$, $QR$, $RS$, $SC$. Or we can proceed directly without the aid of the pentagon, as follows.

With $M$ as center and radius $MO$ draw a circle cutting $CM$ at $H$. See the second diagram in Fig. 54. The length $CH$ is the side of the inscribed regular decagon. For $CH = CJ = m$, and $J$ is that point on CO which divides the radius into two segments such that the longer segment is the mean proportional between the whole radius and the shorter segment; and we shall see that this longer segment $m$ is the side of the inscribed regular decagon.*

For $m + r = \frac{r}{m}$ (from the similar triangles $KCO$ and $OCH$); subtracting 1 from each side gives us

$$m = \frac{r-m}{r}.$$  

This division of the radius $r$ into two parts $m$ and $r - m$ such that $m$ is the mean proportional between $r$ and $r - m$ is known as "The Golden Section." The radius $AO$ in Fig. 55 is divided in this way, but so that the order of the parts $r - m$ and $m$ is the opposite of their order in Fig. 54. If now we make $AF$ (Fig. 55) equal to $m$ and draw $EF$, the two triangles $AEF$ and $AFO$ will be similar (by Principle 5); for they have angle $FAO$ in common and the sides including this angle proportional. Since triangle $AFO$ is isosceles, triangle $AEF$ must also, and $AF = FE = EO$. It follows that $LFAO = 2(LFOA)$, and the sum of the angles of

*To THE TEACHER: The next paragraph provides the proof for this decagon construction. The proof for the pentagon construction is derived therefrom algebraically. Omit this proof if the students have not enough facility in algebra to follow it.
triangle $AFO$ equals $5(LFOA)$. Consequently $LFOA = \frac{1}{5}$ of $180^\circ = \frac{1}{3}$ of $360^\circ$, and $AF$ is one side of the inscribed regular decagon.

Since $m^2 + rm - r^2 = 0$ (Why?) we can find $m$, the side of the decagon, in terms of $r$. We get $m = \frac{-r \pm \sqrt{r^2 + 4r^2}}{2} = \frac{r}{2}(-1 \pm \sqrt{5})$; or, choosing the plus sign so that $m$ shall be positive, we have $m = \frac{r}{2}(\sqrt{5} - 1) = CH = CJ = CT$ (Fig. 54).

We wish now to find the length of the side $2x$ (Fig. 56) of the inscribed regular pentagon, where $m$ is still the side of the decagon. Since $AF$ is the mean proportional between $AB$ and $AG$ (Why?) we have

1. $m^2 = 2r \cdot y$.
2. $x^2 = r^2 - (r - y)^2$ (Why?)

and we have only to eliminate $y$ between these two equations.

From (1) we get $y = \frac{m^2}{2r} = \frac{r^2}{4} (6 - 2\sqrt{5})$.

whence $r - y = r - \frac{r}{4} (3 - \sqrt{5}) = r (\frac{1 + \sqrt{5}}{4})$.

Substituting this value for $r - y$ in (2) we have

$x^2 = r^2 - r^2 \left(\frac{9 + 2\sqrt{5}}{16}\right) = \frac{r^2}{16} (10 - 2\sqrt{5})$ and $x = \frac{r}{4} \sqrt{10 - 2\sqrt{5}}$.

Consequently the side $2x$ of the inscribed regular pentagon is $\frac{r}{2} \sqrt{10 - 2\sqrt{5}}$. Now compute the length $CE$ yielded by our original construction of the inscribed regular pentagon and show that this, too, is equal to $\frac{r}{2} \sqrt{10 - 2\sqrt{5}}$.

There is an interesting relation between the sides of the pentagon and decagon in Fig. 54, namely, that $CP$, $CT$, $CH$, and $CE$ are all equal.
and \( r \) are equal in length to the sides of a right triangle; that is, \((CP)^2 = m^2 + r^2\). Satisfy yourself that this is so. Also show that \( OE = m \) and notice that this interesting Pythagorean relationship is exemplified in triangle \( CEO \). Thus our original construction of the pentagon involved the construction of \( m \) when we used the radius \( MC \), equal to \( \frac{r}{2} + m \), in order to locate \( E \).

To inscribe a regular polygon of fifteen sides in a given circle.

GIVEN: Circle with center 0 (Fig. 57).

TO CONSTRUCT: A regular polygon of fifteen sides with its vertices on the circle.

CONSTRUCTION: Construct one side \( AC \) of an inscribed regular hexagon and also one side \( AB \) of an inscribed regular deca-

Fig. 57

We have shown how to construct inscribed regular polygons of 3, 6, 12, \ldots; 4, 8, 16, \ldots; 5, 10, 20, \ldots; and 15, 30, 60, \ldots sides. It is impossible by straightedge and compasses to construct inscribed regular polygons of 7, 9, 11, 13, or 19 sides, or any multiples of these. The in-

dscribed regular polygon of 17 sides can be constructed by straightedge and compasses, but the method is so long as to be impracticable.

By means of these constructions we are able with straightedge and compasses to draw angles of 120°, 60°, 30°, 15°, 7\frac{1}{2}°, \ldots; 90°, 45°, 22\frac{1}{2}°, 11\frac{1}{2}°, \ldots; 72°, 36°, 18°, 9°, \ldots; and by subtracting an angle of 15° from an angle of 18° we can construct an angle of 3°. In general,
however, we cannot trisect an angle by straightedge and compasses, and so are unable to construct angles of \(1^\circ\) or \(2^\circ\). The proof of this cannot be given here.

**EXERCISES**

Make each of the following constructions with straightedge and compasses.

1. Circumscribe a regular hexagon about a circle.
2. Circumscribe a regular octagon about a circle.
3. Construct an angle of \(75^\circ\).
4. Construct an angle of \(6^\circ\).
5. Construct an angle of \(54^\circ\).
6. How many degrees are there in each angle of a regular pentagon?
7. Construct a regular pentagon, given one side \(AB\).
8. Construct a regular octagon, given one side \(AB\).
9. Construct a regular decagon, given one side \(AB\).
10. On a given line segment as side, construct a polygon similar to a given polygon.
11. In a given equilateral triangle inscribe three equal circles, so that each circle shall be tangent to the other two and tangent also to two sides of the triangle.
12. In a given square inscribe four equal circles, each tangent to two others and to two sides of the square.
13. In a given square inscribe four equal circles, each tangent to two others and to one side and one side only of the square.
14. Construct a five-pointed star by extending the sides of a regular pentagon. Draw the diagonals of the pentagon also.
15. The development of the regular twelve-sided solid (dodecahedron) is shown in Fig. 58; each of the twelve faces is a regular pentagon. Make a tracing of Fig. 58. On this tracing indicate the edges of the figure on which there should be flaps suitable for pasting.

16. Construct a regular pentagon, given one of the diagonals.

17. Draw a tangent to a given circle such that the part of the tangent which lies between the point of contact and a given line shall have a given length $l$. See Fig. 59.
In the picture above, the boy [photographing the bassoon player has a camera with an adjustable diaphragm controlling the amount of light that is admitted to the camera. The opening of the diaphragm is a regular polygon of so many sides that it approximates a circle. The diaphragm can be adjusted to make this polygon large or small, as shown in the inserts at the bottom of the picture. By doubling the diameter of the opening of the diaphragm, four times as much light is admitted to the camera. This illustrates a principle concerning area that is considered in Chapter 7.
CHAPTER 7

Al'ea and Length

IN CHAPTER 2 we discussed line measure and from then on referred frequently to lengths of lines and to equal and unequal lengths. However, line measure is only one of several sorts of measure frequently needed. Surface measure, or area, is a second sort.

You are familiar with the principles that are used in finding the areas of various plane figures, including the parallelogram, triangle, trapezoid, and circle. You will find these principles repeated here as theorems, together with a few other principles which may be new to you. However, the primary purpose here will be to study the logical development of these principles as theorems; that is, to see upon what definitions and assumptions these theorems depend and how one may be derived from another.

Since we are bringing a new idea or concept into our logical structure, we shall need at least one new definition and possibly one or two new assumptions. There are several different ways of beginning. In the present development we shall use the ideas of area that you have learned from your previous work in mathematics and shall see how these can be developed logically.*

*An entirely different way of beginning this development is discussed in the "Note on the Area of Polygons" at the end of this chapter, page 222.
Below we list two assumptions and a definition that we require.

**Area Assumption 1.** Every polygon has a number, called its area, such that
(a) equal polygons have equal areas, and
(b) the area of a polygon is equal to the sum of the areas of its constituent polygons.

*The area of a square each side of which is one unit in length shall be the unit of area.* We shall assign the number 1 as the area of this unit square.

**Area Assumption 2.** The area of a rectangle is equal to the product of its length times its width. That is, \( A = bh \). This is true even when one, or both, of the dimensions of the rectangle is an irrational number.

If we had used the alternative approach suggested on page 222, we could have avoided making these two assumptions and could have proved both of them as theorems, but only after a great deal of detailed work.

**EXERCISES**

1. To find the area of Fig. 1, you must think of the figure as being divided into several rectangles. You can find the area of each of these rectangles if you know its length and its width. See if you can use the coordinates of the vertices of the figure to find these dimensions and thus find the area of the entire figure. If you cannot do this in your head without marking the book, copy the figure on blank paper.
2. Check your computation of the area you found for Fig. 1 on page 199. Do this by making a scale drawing of the figure on squared paper and counting the number of squares included within its boundary.

**Theorem 25.** The area of a right triangle is equal to half the product of one side times the altitude upon that side.

![Diagram of a right triangle](image)

**GIVEN:** Triangle $ABC$ (Fig. 2) in which $\angle C = 90^\circ$.

**TO PROVE:** Area of $\triangle ABC = \frac{1}{2}AC \times BC = \frac{1}{2}bh$.

**ANALYSIS:** Since we know how to find the area of rectangles only, it is clear that we must in some way relate this problem to a rectangle. We can do this by drawing parallels to $AC$ and $BC$ through $B$ and $A$ respectively. These parallels meet at $D$. The area of the rectangle thus formed is $bh$, and we can prove that $\triangle ADB = \triangle ABC$.

Give the complete proof.

**EXERCISES**

1. Show that the area of $\triangle ABC$ (Fig. 2) equals $\frac{1}{2}AB \times CE$ if $CE$ is perpendicular to $AB$. **Suggestion:** Consider the similar triangles $ABC$ and $CBE$.

2. Now show by the following method that the area of $\triangle ABC$ (Fig. 2) equals $\frac{1}{2}AB \times CE$. First construct a rectangle on $AB$ as one side and with altitude equal to $CE$. Then make use of the fact that the area of a polygon is equal to the sum of the areas of its constituent polygons.

*The symbol $\triangle$ means "triangle."*
**Theorem 26.** The area of any triangle is equal to half the product of one side times the altitude upon that side. That is, \( A = \frac{1}{2}bh \).

![Fig. 3](image)

**GIVEN:** \( \triangle ABC \) (Fig. 3) in which \( BD \) is perpendicular to \( AC \).

**TO PROVE:** Area of \( \triangle ABC = \frac{1}{2}AC \times BD = \frac{1}{2}bh \).

**PROOF:** Area of \( \triangle ABD = \frac{1}{2}mh \)

Area of \( \triangle CBD = \frac{1}{2}nh \)

Area of \( \triangle ABC = \frac{1}{2}(m+n)h \) if \( D \) is between \( A \) and \( C \);

\[ = \frac{1}{2}(m-n)h \] if \( C \) is between \( A \) and \( D \);

\[ = \frac{1}{2}bh \] in either case.

**Corollary 26a.** The area of a parallelogram is equal to the product of one side times the altitude upon that side. That is, \( A = bh \).

Any parallelogram can be regarded as the sum of two triangles having equal bases and altitudes. See Fig. 4.

**Fig. 4**

Give the complete proof.
Corollary 26b. The area of a trapezoid is equal to half the product of the sum of its bases times its altitude. That is, \( A = \frac{1}{2}(b+b')h \).

Give the complete proof. See Fig. 5.

EXERCISES

1. Find the area of the diagram in Fig. 6.
2. Find the area of the two triangles \( ABC \) in Fig. 3 on page 201, first making the necessary measurements.
3. Check your results in Ex. 2 as follows: In each triangle measure another side and its altitude and find the areas from these measurements.
4. Find the area of the parallelogram in Fig. 4 on page 201.
5. Find the area of the trapezoid in Fig. 5.
6. Find the area of an equilateral triangle of side \( s \). Leave your answer in radical form.
7. Each face of a regular triangular pyramid (often called a regular tetrahedron) is an equilateral triangle. Find the total area of this pyramid in terms of the length \( e \) of one edge. See Fig. 7.
8. How would you compute the total area of a triangular prism?
9. Prove that the areas of two similar triangles are to each other as the squares of any two corresponding sides.
*10. Prove that if two triangles have an angle in common, their areas are to each other as the products of the sides including this common angle. That is, in Fig. 8 prove that
\[
\frac{\text{Area } ABC}{\text{Area } ADE} = \frac{AB \times AC}{AD \times AE}
\]

*Suggestion: Draw CD and compare first the areas of triangles ABC and ADC, and then the areas of triangles ADC and ADE.

The area of any polygon can be found by dividing the polygon into triangles by means of lines radiating from a point and then measuring the sides and altitudes of these triangles. See Fig. 9. The point need not be inside the polygon or at a vertex, though these are the most convenient positions to choose.

The area of a polygon can be found approximately by placing a rectangular network upon the polygon and counting the number of squares inside the polygon. See Fig. 10. The smaller the squares the more nearly will this number approach the true area of the polygon.
In similar manner we can find approximately the area of any convex closed curve, that is, any closed curve that bulges outward at every point.*

A regular polygon of $n$ sides can be divided into $n$ isosceles triangles by drawing lines from the vertices to the center of the inscribed circle. See Fig. 11.

*The radius of the inscribed circle is sometimes called the apothem of the regular polygon, and the radius of the circumscribed circle is sometimes called the radius of the regular polygon.

**EXERCISES**

1. Prove that the area of a regular polygon is equal to half the product of its perimeter times its apothem. That is, $A = \frac{1}{2}pa$, where $p=ns$. See Fig. 11.

2. A regular hexagon is 5 inches on a side. Find its apothem and area.

3. What is the area of a regular hexagon inscribed in a circle of radius $r$?

4. Find the perimeter and area of an equilateral triangle inscribed in a circle of radius $r$.

5. An equilateral triangle is $s$ inches on a side. Find its apothem and radius.

6. In Chapter 6 (page 193) the side of a regular pentagon of radius $r$ was given as $\frac{r\sqrt{10 - 2\sqrt{5}}}{2}$. Find the area of such a figure. Answer: $\frac{5}{8}r\sqrt{10 + 2\sqrt{5}}$, or about $2.4r^2$.

*See the "Note on Approximate Areas by Means of Square Networks" at the end of this chapter, pages 225 and 226.
7. The two polygons in Fig. 9 on page 203 are equal. Find the area of each of these polygons, first making the necessary measurements in millimeters.

8. Find the approximate area of the polygon in Fig. 10 on page 203 by counting squares. The lines of the network are a millimeter apart. Compare your result with the areas obtained in Ex. 7. The three polygons in Figs. 9 and 10 are all equal.

9. Find the approximate area of the irregular figure in Fig. 12 by counting the small squares that lie inside.

*10. Prove that the perimeters of two regular polygons of the same number of sides are to each other as their radii.

*11. Prove that the areas of two regular polygons of the same number of sides are to each other as the squares of their radii.

Theorem 27. The areas of two similar polygons are to each other as the squares of any two corresponding sides.

GIVEN: The similar polygons $ABCDE$ and $A'B'C'D'E'$ (Fig. 13).
TO PROVE: \[ \frac{\text{Area } ABCDE}{\text{Area } A'B'C'D'E'} = \frac{(AB)^2}{(A'B')^2} \]

ANALYSIS: We have already proved (Ex. 9, page 202) that the areas of two similar triangles are to each other as the squares of any two corresponding sides. We need to show that the areas of the polygons are to each other as the areas of two corresponding triangles, for example, \(ABC\) and \(A'B'C'\).

PROOF: \[ \frac{\triangle ABC}{\triangle A'B'C'} = \frac{(AB)^2}{(A'B')^2} \quad \frac{\triangle ACD}{\triangle A'C'D'} = \frac{(AC)^2}{(A'C')^2} \quad \frac{\triangle ADE}{\triangle A'D'E'} = \frac{(AD)^2}{(A'D')^2} \cdot \frac{\triangle MBC}{\triangle M'B'C'} \quad \frac{\triangle MCD}{\triangle M'C'D'} \quad \frac{\triangle MDE}{\triangle M'D'E'} \]

Then \( \triangle ABC = k \cdot \triangle A'B'C' \)
\( \triangle ACD = k \cdot \triangle A'C'D' \)
\( \triangle ADE = k \cdot \triangle A'D'E' \)
\( \triangle ABC + \triangle ACD + \triangle ADE = k \cdot (\triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E') \)
or
\( \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle A'B'C' + \triangle A'C'D' + \triangle A'D'E'} = \frac{k}{(AB)^2} \cdot \frac{(AB)^2}{(A'B')^2} \cdot \frac{(A'B')^2}{(A'B')^2} \cdot \frac{\text{Area } ABCDE}{\text{Area } A'B'C'D'E'} = \frac{(AB)^2}{(A'B')^2} \]

The unusual algebraic treatment of the continued proportion used above is applicable to any proportion. That is, if \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \), we can always show that \( \frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{a}{b} \) by letting \( \frac{a}{b} = k \cdot \frac{c}{d} \) and so forth. See Exercises 37 and 38, page 66.

By means of Theorem 27 we can compare the areas of similar solids, as follows: The areas of two similar prisms, pyramids, cylinders, or cones are to each other as the squares of any two corresponding dimensions. In similar manner it can be proved that the volumes of two similar prisms, pyramids, cylinders, or cones are to each other as the cubes of any two corresponding dimensions.
EXERCISES

1. An architect submits a design for a memorial window drawn to the scale of 1 to 20. Compare the area of the window as drawn with the area of the actual window.

2. Helen Thurman has just finished making a hooked rug. Now she wants to make another with the same proportions. The new rug is to be \(1\frac{1}{3}\) times as long as the one she has just completed. She will use about how many times as much yarn for the larger rug as she used for the smaller one?

3. Prove that if similar polygons are constructed on the three sides of a right triangle, the area of the largest will equal the sum of the areas of the other two. See Fig. 14.

4. I wish to construct a cube whose volume will be double the volume of a given cube. How long must I make each edge as compared with an edge of the given cube?

5. Compare the areas of the two cubes of Ex. 4.

6. Packer's soda fountain sells ice-cold lemonade in cone-shaped paper cups. The 5-cent size is 4 inches deep, and the 10-cent size is 5 inches deep. Both sizes of cup have the same angle at the vertex. Assuming that the cups are filled to the brim, which is the better buy: one 10-cent cup of lemonade or two 5-cent cups?

7. The 40-foot cylindrical water tower at Salem is to be replaced by a 55-foot tower having the same shape and proportions as the present one. Specifications call for the same sort and thickness of sheet iron in the new tank as in the old. By what per cent will the amount of sheet iron in the water tower be increased? By what per cent will the capacity of the water tower be increased?
*8. Fig. 15 suggests an alternative proof for the Pythagorean Theorem. This proof is based on the idea of area. Notice that the square on the hypotenuse of one of the right triangles is equal to the outside square minus the four right triangles. Complete the proof by showing that this is equal also to the sum of the squares on the other two sides of the right triangle.

![Fig. 15](image1)

![Fig. 16](image2)

*9. Fig. 16 suggests another proof for the Pythagorean Theorem, also based on the idea of area. See if you can prove it this way.

CIRCUMFERENCE OF A CIRCLE

It is not hard to see how the idea of area for figures bounded by straight lines can be extended so that it may apply to closed curvilinear figures. In fact, we have already considered the area of an irregular curvilinear figure. See Ex. 9, page 205. However, it is very much more difficult to extend the idea of line measure so that it applies to curved lines. Indeed, it is difficult to state precisely what we mean by the length of a curved line. This is especially true if the line is a wavy line. In discussing the length of curved figures, therefore, we shall limit ourselves to the simplest case of all, the circumference of a circle.

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For a number of years you have used the term "circumference of a circle" without defining it precisely. It can be defined as follows. The more points of a circle you choose to join successively by chords, the greater will be the length of the resulting broken line (by Corollary 12c). See Fig. 17. But there is an upper limit to this length, as is shown in the note at the end of this chapter. This upper limit is called the CIRCUMFERENCE OF THE CIRCLE.

Similarly, the LENGTH OF AN ARC OF A CIRCLE is the upper limit of the length of a series of inscribed broken lines, and is the same proportional part of the circumference that its central angle is of four right angles. It follows, therefore, that the lengths of two arcs of the same circle have the same ratio as their corresponding central angles. For example, in Fig. 18,

\[
\begin{align*}
\text{Arc } AB &= 54. \\
\text{Circumference } &= 360' \\
\text{Arc } BC &= 21 \\
\text{Circumference } &= 360; \\
\frac{\text{Arc } AB}{\text{Arc } BC} &= \frac{54}{21} = 18 \frac{6}{7}.
\end{align*}
\]

In particular, equal arcs have equal lengths. See the first paragraph on page 135.

If we wish, we can make a distinction with respect to sign between the lengths of the directed arcs \(AB\) and \(BA\) similar to the distinction between the directed central angles \(AOB\) and \(BOA\). See page 49. Only rarely, however, shall we find such a distinction helpful.

*See the "Note on the Circumference and Area of a Circle," page 223.
A regular polygon is inscribed in a circle. If, now, another regular polygon of double the number of sides is inscribed in the circle, its area will be greater than the area of the first polygon. For the second polygon is made up of the first polygon plus several isosceles triangles. As the number of sides of the inscribed polygon is increased, its area increases also; but not without limit, for it can never be as great as the area of any circumscribed polygon. See Fig. 19. The number which the areas of the inscribed polygons approach as their limit as the number of their sides is indefinitely increased is called the **area of the circle**.

**The Perimeters of Inscribed Regular Polygons**

An inscribed regular polygon of \( n \) sides has each side of length \( s \).

Let us find \( s' \), the length of each side of a regular polygon of \( 2n \) sides inscribed in the same circle.

In Fig. 20, \( AF \) is the mean proportional between \( AB \) and \( AG \). Why? Therefore we have

1. \((s')^2 = 2r \cdot m\).

Moreover,

2. \((r - m) = \sqrt{r^2 - \left(\frac{s'}{2}\right)^2}\). Why?

*See the "Note on the Circumference and Area of a Circle" at the end of this chapter, page 223.
We have only to eliminate \( m \) between these two equations.

From equation (1) we get
\[
m = \frac{(s_a)^2}{2r},
\]
and from equation (2),
\[
m = r - \sqrt{r^2 - \left(\frac{a}{2}\right)^2}.
\]
Therefore, \((s_a)^2 = 2r^2 - r\sqrt{4r^2 - (s_a)^2},\)
or, \(s_a = \sqrt{2r^2 - r\sqrt{4r^2 - (s_a)^2}}.\)

For example, if \( n = 3 \), then \( s_3 = r\sqrt{3}. \) Then \( s_6 \) should equal
\[
\sqrt{2r^2 - r\sqrt{4r^2 - 3r^2}},
\]
or, \( r \), which is correct.

By means of this formula for \( s_n \) in terms of \( r \) and \( s_4 \), we can find the perimeters of inscribed regular polygons of 8, 16, 32, 64, . . . sides. The side \( s_4 \) of an inscribed square is \( r\sqrt{2}, \) and its perimeter \( P_4 \) is \( 4r\sqrt{2}. \)

Therefore \( s_8 = \sqrt{2r^2 - r\sqrt{4r^2 - 2r^2}} = \sqrt{2r^2 - r\sqrt{2}} = r\sqrt{2 - \sqrt{2}} \),
and \( P_8 = 8r\sqrt{2 - \sqrt{2}}. \)

Similarly, \( s_{16} = \sqrt{2r^2 - r\sqrt{4r^2 - r^2(2 - \sqrt{2})}} = \sqrt{2r^2 - r\sqrt{2 + \sqrt{2}}} = r\sqrt{2 - \sqrt{2 + \sqrt{2}}}, \)
and \( P_{16} = 16r\sqrt{2 - \sqrt{2 + \sqrt{2}}}. \)

Verify (at least approximately) the computations indicated in the following table.

\[
\begin{array}{|c|c|c|}
\hline
n & s_n & P_n = ns_n \\
\hline
4 & r\sqrt{2} = 1.4142r & 5.6568r \\
8 & r\sqrt{2 - \sqrt{2}} = 0.7654r & 6.1232r \\
16 & r\sqrt{2 - \sqrt{2 + \sqrt{2}}} = 0.3901r & 6.2416r \\
32 & r\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}} = 0.1960r & 6.2720r \\
\hline
\end{array}
\]
The values for $p_3, p_5, p_{17}, \ldots$ show that the perimeters of inscribed regular polygons increase as the number of their sides is doubled and redoubled. But the amount of increase grows continually less and $P_n$ approaches the circumference of the circle as its limit. By computing more and more of these perimeters, at the same time carrying our computations to more and more decimal places, the circumference of a circle can be shown to be 6.28318 ... times its radius, or 3.14159 ... times its diameter. This latter number is a non-ending decimal and is commonly denoted by the Greek letter $\pi$. The circumference is equal, therefore, to $2\pi$ times the radius. That is, $c = 2\pi r$.

The value $\frac{22}{7}$ for $\pi$ is often accurate enough for the job in hand. Find to four decimal places the error you make when you use this value for $\pi$.

We have proved that the area of any regular polygon may be expressed as $A = \frac{1}{2}pa$ (Ex. 1, page 204). We have previously defined the area of a circle as the limit of the areas of inscribed regular polygons as the number of their sides is indefinitely increased. That is, $A$ is the limit of $\frac{1}{2}pa$ as $n$ increases indefinitely. See Fig. 21.

Since $a = \sqrt{r^2 - \left(\frac{s}{2}\right)^2}$, $\frac{1}{2}pa = \frac{1}{2}p\sqrt{r^2 - \left(\frac{s}{2}\right)^2}$.

As $n$ increases indefinitely, the limit of $P$ is $c$ and the limit of $s$ is zero. Therefore the limit of $\frac{1}{2}pa$ is $\frac{1}{2}cr$, or $A = \frac{1}{2}cr$. But $c = 2\pi r$; so we may write $A = \pi r^2$.

*We have taken for granted here that the limit of the product of two variables, $p$ and $\sqrt{r^2 - \left(\frac{s}{2}\right)^2}$, is equal to the product of their limits.
The circumferences of two circles are to each other as their radii. For \( c_1 = 2\pi r_1 \) and \( c_2 = 2\pi r_2 \), and so

Similarly, the areas of two circles are to each other as the squares of their radii.

\[
\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}
\]

The figure bounded by two radii and an arc of a circle is called a **sector** of the circle.

If the angle of the sector is \( m^\circ \) and the length of the arc is \( s \) (Fig. 22), then

\[
s = \frac{m}{360} \cdot 2\pi r = \frac{m}{360} \cdot \pi r^2.
\]

The area of the sector is

\[
\frac{m}{360} \cdot \pi r^2 = \frac{1}{2} rs.
\]

**EX ER C I SE S**

1. Three circles have the following radii: 7.24 inches, 9 feet 4 inches, 3.5 centimeters. Find the circumference of each circle.

2. Find the areas of the circles in Ex. 1.

3. Find the radii of circles with the following circumferences: 7.152 inches, 90.2 centimeters.

4. Find the radii of circles with the following areas: 19.46 square feet, 281 square centimeters.

5. Find the areas of circles with the following circumferences: 15.2 feet, 2 feet 3 inches, cinches.

6. Find the circumferences of circles with the following areas: 604 square centimeters, 19 square feet, 84 square inches.
7. Prove that in two circles of radii \( r \) and \( kr \), chords of lengths \( c \) and \( hc \) respectively have arcs of lengths \( s \) and \( ks \) respectively. See Fig. 23.

8. Prove that the areas of two circles are to each other as the squares of their circumferences.

9. The diameters of two circles are 3 and 6. How many times is the area of the first contained in the area of the second?

10. What is the area of the square inscribed in a circle of radius 5 inches?

11. Two regular hexagons have sides 2 and 3 respectively. How many times is the area of the first contained in the area of the second?

12. Two regular polygons of 51 sides have areas 8 and 9 respectively. How many times is the perimeter of the first polygon contained in the perimeter of the second?

13. The area of the cross-section of a heavy wooden plank 2 inches \( \bar{F} \) inches thick is 20 square inches. Another plank has the same proportions but is 1\( \frac{1}{6} \) inches thick. What is the area of the cross-section of the second plank? See Fig. 24.

14. The circumference of one circle is twice that of another. How many times is the area of the smaller circle contained in the area of the larger circle?

15. The area of one circle is five times that of another. How many times will the circumference of the first contain the circumference of the second?
16. Find the difference between the area of a circle of radius \( r \) and (1) the area of the circumscribed square; (2) the area of the inscribed square. See Fig. 25.

17. Find the total area of a cylinder 7 inches tall and 3 inches in diameter.

18. How long a straw could you fit (without bending it) into the cylinder described in Ex. 17?

19. A sector of 40° is cut from a circle of radius 6. Find the area of the sector.

20. What is the length of the arc of the sector in Ex. 19?

21. Two equal parallel chords are drawn 8 inches apart in a circle of 8-inch radius. Find the area of that part of the circle which lies between the chords.

22. Prove that if semicircles are constructed on the three sides of a right triangle, the area of the largest will equal the sum of the areas of the other two. Prove also that the area of the triangle will equal the sum of the areas of the two curvilinear figures that are shaded in Fig. 26.

23. From a circular log 24 inches in diameter a man wishes to cut a pillar so that its cross-section shall be as large a regular octagon as possible. Find the length of each side of the octagon.

24. Find the perimeter of a regular dodecagon inscribed in a circle of radius \( r \). How much shorter is it than the circumference of the circle?

25. Find the perimeter of a regular heptagon inscribed in a circle of radius 5. (Use trigonometric tables.)
26. The cylindrical silo that is shown in Fig. 27 has a conical roof. The circumference of the silo is 52 feet. The slant height of the roof, after the J-foot overhang at the eaves has been deducted, is 11 feet 2 inches. Find the angle at the vertex of the roof. (Use trigonometric tables.)

Fig. 27  Fig. 28  Fig. 29  Fig. 30  Fig. 31

27. Show that the lateral area of a right prism is equal to the perimeter of the base times the altitude. See Fig. 28.

28. Show that the lateral area of a right cylinder is equal to the circumference of the base times the altitude. See Fig. 29.

29. Show that the lateral area of a regular pyramid is equal to half the perimeter of the base times the slant height. That is, \( A = \frac{1}{2} pl \). See Fig. 30.

30. Show that the lateral area of a right cone is equal to half the circumference of the base times the slant height. That is, \( A = \pi rl \). See Fig. 31.

The following formulas are also useful.

The volume of any prism is equal to the area of the base times the altitude. That is, \( V = Bh \).

The volume of a cylinder is equal to the area of the base times the altitude. If the base is a circle, \( V = \pi r^2 h \).

The volume of any pyramid is equal to one-third the area of the base times the altitude. That is, \( V = \frac{1}{3} Bh \).
The volume of a cone is equal to one-third the area of the base times the altitude. If the base is a circle, \( V = \frac{1}{3} \pi r^2 h \).

The area of a sphere equals \( 4 \pi r^2 \). That is, \( A = 4 \pi r^2 \).

The volume of a sphere equals one-third the area of its surface times the radius. That is, \( V = \frac{1}{3} \pi r^3 \).

31. Using only straightedge and compasses, inscribe in a given circle a regular polygon similar to a given regular polygon.

32. Using only straightedge and compasses, divide the area of a given circle into two equivalent parts by means of a circle concentric with the given circle. See Fig. 32.

33. Using only straightedge and compasses, divide the area of a given circle of radius \( r \) into three equivalent parts by means of two circles concentric with the given circle.

34. The area of a given circle of radius \( r \) is to be divided into \( n \) equivalent parts by means of circles concentric with the given circle. Find the radii of the inner circles in terms of \( r \).

35. Explain how you would construct a polygon similar to a given irregular polygon and having \( n \) times the area.

36. Explain how to construct a circle whose area shall be to the area of a given circle as \( m \) is to \( n \).

37. Two spheres have their radii in the ratio 1 to 2. Compare their volumes.

38. Two spheres have their volumes in the ratio 1 to 2. Compare their radii.

39. Spheres concentric with a given sphere divide its volume into \( n \) equivalent parts. Find the radii of these concentric spheres.
MAXIMA AND MINIMA

The following theorems, stated here without proof, have many interesting applications.

1. Of all the polygons with \( n \) given sides, that one which can be inscribed in a circle has the greatest area. Thus in Fig. 33, of all quadrilaterals having their sides equal respectively to the sides of quadrilateral I, quadrilateral II has the greatest area.

![Fig. 33](image)

The truth of this theorem is obvious in the case of parallelograms. For of all parallelograms with given sides, only the rectangle can be inscribed in a circle, and it is obvious from Fig. 34 that this rectangle has the greatest area.

![Fig. 34](image)

2. Of all polygons having equal perimeters and the same number of sides, the polygon of greatest area will be equilateral. It follows from the previous theorem that this equilateral polygon of greatest area can be inscribed in a circle, and hence is also a regular polygon. For example, of all quadrilaterals having a perimeter of ten inches, the square has the maximum area.

You have 700 feet of fencing with which to enclose a four-sided garden. What will be the most economical shape for your garden?

3. Of several regular polygons having the same perimeter, that one which has the greatest number of sides has the greatest area. Thus if an equilateral triangle, a square, and a regular pentagon all have the same perimeter, the regular pentagon will have the greatest area. From this we infer that of all plane figures having the same perimeter, the circle encloses the greatest area.
4. The previous theorem can be restated as follows: Of several regular polygons having the same area, that one having the greatest number of sides has the least perimeter.

Why is a circular cross-section of pipe—that is, a perfectly round pipe—the most advantageous shape to use for water pipes?

How does pinching the outer end of the exhaust pipe of an automobile affect the exhaust of the engine?

5. In three dimensions, of all solids enclosing a given volume the sphere has the least area.

If your young brother buys a candy sucker and wants it to last as long as possible, why should he buy a spherical one instead of a cubical one of the same volume?

**Euler's Theorem**

Another interesting theorem is Euler's Theorem, which gives the relation between the number of faces, vertices, and edges of any polyhedron, that is, of any solid having only plane faces. His formula, \( f + v = e + 2 \), states that the number of faces plus the number of vertices is equal to the number of edges plus 2. It is easy to verify this for the cube and other simple solids. Apply it to a house with several gables.

**The Regular Polyhedrons**

At each vertex of a cube there are three right angles. There cannot be any other regular solid having squares for faces, for if so, then four faces would have a common vertex with four right angles at that point, and the corner would be flattened down until the vertex ceased to exist. The regular tetrahedron (triangular pyramid) has three equilateral triangles meeting at each vertex. Would it...
be possible for some other regular polyhedron to have four equilateral triangles meeting at each vertex? Five? Six!

Could there conceivably be a regular polyhedron in which three regular pentagons meet at each vertex? More than three? Three regular hexagons?

There cannot be more than five convex regular polyhedrons. What are they? What sorts of faces do they have? In each of these polyhedrons how many faces meet at each vertex?

**SUMMARY**

*Area Assumption 1.* Every polygon has a number, called its area, such that
(a) equal polygons have equal areas, and
(b) the area of a polygon is equal to the sum of the areas of its constituent polygons.

**DEFINITION:** The area of a unit square shall be 1.

*Area Assumption 2.* The area of a rectangle is equal to the product of its length times its width. That is, \( A = bh \).

25. The area of a right triangle is equal to half the product of one side times the altitude upon that side.

26. The area of any triangle is equal to half the product of one side times the altitude upon that side. That is, \( A = \frac{1}{2}bh \).

26a. The area of a parallelogram is equal to the product of one side times the altitude upon that side. That is, \( A = bh \).

26b. The area of a trapezoid is equal to half the product of the sum of its bases times its altitude. That is, \( A = \frac{1}{2}(b+b')h \).

**DEFINITIONS:** apothem and radius of a regular polygon 220
27. The areas of two similar polygons are to each other as the squares of any two corresponding sides.

Definitions: circumference and length of arc of a circle, area of a circle, $\pi$, sector

Review Exercises

1. You are given only the mid-points of the sides of a triangle. How do you find the vertices?

2. Prove that the line joining the feet of the altitudes on the two equal sides of an isosceles triangle is parallel to the third side.

3. Prove that the sum of the angles at the outer points of a regular five-pointed star is equal to 180°.

4. If two opposite angles of a quadrilateral are right angles, what relation exists between the bisectors of the other two angles? Prove that your statement is correct.

5. Prove that the line joining the mid-points of the diagonals of a trapezoid is parallel to the bases.

*6. Prove that the line joining the mid-points of the bases of a trapezoid passes through the intersection of the non-parallel sides extended.

*7. Prove that the diagonals of a trapezoid intersect on the line joining the mid-points of the bases.

*8. Combine the theorems in Exercises 6 and 7 in one statement. Make a careful drawing to exemplify it.

9. If $AB$ and $CD$ are equal parallel chords in a circle, prove that $AD$ and $BC$ are diameters.

See Fig. 35.

10. Prove that the sum of the perpendiculars drawn from any point within a regular polygon to its $n$ sides is equal to $n$ times its apothem.

Fig. 35
NOTE ON THE AREA OF POLYGONS

It is possible to avoid Area Assumptions 1 and 2 that were introduced at the beginning of this chapter if we derive the idea of area directly from the idea of similar triangles and proportion. The content of Assumptions 1 and 2 will then appear as theorems that we can prove.

---

NOTE ON THE CIRCUMFERENCE AND AREA OF A CIRCLE

It can be proved that the perimeter of an inscribed irregular polygon of $n$ sides is less than the perimeter of the inscribed regular polygon of $n$ sides. It can also be proved that the perimeter of a circumscribed irregular polygon of $n$ sides is greater than the perimeter of the circumscribed regular polygon of $n$ sides. It is evident that the perimeter of the inscribed regular polygon of $n$ sides is less than the perimeter of the circumscribed regular polygon of $n$ sides and that the ratio of these perimeters will be equal to $\frac{AM}{AP}$ or $\frac{OM}{r}$ (Fig. 38). As $n$ increases indefinitely, $OM$ approaches $r$ as its limit, and the ratio $\frac{OM}{r}$ has the limit 1. This means that the perimeters of pairs of inscribed and circumscribed regular polygons of the same number of sides approach the same limit as the number of sides is increased indefinitely. This is the limit also of the perimeters of inscribed and circumscribed irregular polygons where all the sides approach zero in length as the number of sides is increased indefinitely. This common limit of the perimeters we call the circumference of the circle.

LENGTH OF $\overline{AB}$. This method of inscribed and circumscribed polygons can be applied also to any arc of a circle. The length of the arc $\overline{AB}$ (Fig. 39) is the common limit approached by the length of those portions of inscribed and circumscribed polygons comprised between the radii $\overline{OA}$ and $\overline{OB}$.

THE CONSTANT RATIO. If irregular polygons of the same number of sides be inscribed in two circles of radii $R$ and $r$ (Fig. 40, page 224), and if then the number of their sides be increased indefinitely, we know that the limit of $(S_1 + S_2 + \ldots + S_k)$ equals $C$ and the limit of $(8_1 + 8_2 + \ldots + 8_k)$ equals $c$, where $C$ and $c$ are the circum-
That is, the ratio of circumference to radius is constant for all circles. It is customary to denote the constant ratio of circumference to diameter by the Greek letter \( \pi \). It follows that the circumference of a circle is equal to \( 2\pi r \).

**The Area of a Circle.** We shall define the area of a circle as the limit of the area of inscribed (regular or irregular) polygons in which every side approaches zero as the number of sides is increased indefinitely.

The area of polygon \( ABC \ldots MN \) in Fig. 41 is at least equal to, and will ordinarily be greater than, \( \frac{1}{2}(AB + BC + \ldots + MN) \cdot a \), where \( a \) is the least distance from the center of the circle to a chord. Moreover, as \( n \) increases indefinitely, the area of the inscribed polygon increases indefinitely, but not without limit, for it can never be greater than the area of any circumscribed polygon. That is, 

\[
\frac{a}{2}(s_1 + s_2 + \ldots + s_n) \leq \text{area of polygon } AB \ldots N < \frac{1}{2}(t_1 + t_2 + \ldots + t_m),
\]

where \( a \leq a_1, a \leq a_2, \ldots, a \leq a_n \). But as \( n \) is indefinitely increased

\*Earlier in this chapter we confined our definition of the area of a circle to the simpler case of regular polygons, and restricted the method by which \( n \) should increase to the method of indefinitely repeated doubling.

\[ \text{The symbol } \leq \text{ means "is less than or equal to."} \]

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the limit of \( a \) is \( r \); the limit of \((81+82+\ldots+8_n)\) is the circumference of the circle, \( c \); and the limit of \((t_1+t_2+\ldots+t_n)\) is likewise \( c \). Since now the area of the polygon \( AB \ldots N \) is between two numbers each of which is approaching \( \frac{1}{2}rc \) as its limit, the limit of the area of the polygon must be \( \frac{1}{2}rc \). This limit, \( \frac{1}{2}rc \), shall be by definition the area of the circle. Since \( c = 2\pi r \), we can write the area of the circle in the form \( A = \pi r^2 \).

The area of the circle could have been defined equally well as the limit of the area of circumscribed polygons. The area of any convex closed curve may be defined similarly as the limit of the area of inscribed (or circumscribed) polygons in which every side approaches zero as the number of sides is increased indefinitely.

**AREA OF A SECTOR.** The treatment of the area of a sector of a circle can be given rigorously after the manner outlined for the treatment of the length of an arc.

**NOTE ON APPROXIMATE AREAS BY MEANS OF SQUARE NETWORKS**

We have shown that the area of a polygon is equal to the sum of the areas of its constituent polygons, and that the area of a convex closed curve is the limit of the areas of inscribed and circumscribed polygons. Often, however, instead of calculating these areas accurately with great pains, we should be content with a good approximation if readily obtainable. We can get such an approximation by drawing a rectangular network—a grid of squares—over the polygon or closed curve and counting the number of squares which have no part outside the figure. See Fig. 42. The
smaller these squares, the better the approximation. Another and
equally good approximation would result from taking every square
any portion of which lies inside the figure.

Notice that the boundary of the polygon in Fig. 42 lies within
an irregular strip of squares, each of these squares being partly in-
side and partly outside the polygon. The two approximations differ by the
total area of the irregular strip of squares containing the boundary of the
polygon. Fig. 43 is an enlargement of a small part of the strip of squares.
This figure suggests that if the squares are allowed to become smaller and
smaller indefinitely—by halving, for example—the total area of the irregular strip of squares approaches
zero as its limit; for the limit of the $n$th power of a proper fraction
in this case $\frac{1}{2}$ as $n$ increases indefinitely is always zero.

Since the area of a convex closed curve is the limit of the areas of
inscribed and circumscribed polygons, it is clear that this area, too,
can be expressed as the common limit of the interior and exterior
approximations resulting from the superposition of a square network.
When a bullet is fired from a rifle, or a baseball is thrown to first base, or a transport plane is in flight, the position of the moving object varies continuously with reference to its starting-point. In Chapter 8 we shall relate this idea to our geometry.
CHAPTER 8

Continuous Variation

In this chapter we shall consider situations in which a point varies continuously along a straight or curved line. We shall also consider comparable situations involving angles. Because of the intimate connection between numbers and the points of a line, however, we cannot discuss the continuous variation of points and angles until we have become more familiar with the properties of different sorts of numbers. In particular, we need to know more about rational numbers and irrational numbers, which together constitute what mathematicians call the system of real numbers.

Any number that can be expressed as the quotient of two integers is called a rational number. For example, \( \frac{5}{1}, \frac{2}{3}, \frac{4}{3}, \frac{-2}{1}, \frac{7}{4} \) are rational numbers. Since it is not possible to divide by zero, the integer 0 may appear only as the numerator, never as the denominator. The positive and negative integers and fractions together with zero constitute the rational numbers.

The rational numbers are ever so close together. For no matter how nearly equal two rational numbers may be, it is always possible to find another rational—for ex-
ample, their arithmetic mean—that lies between them. Since we also think of the points of a line as being ever so close together, we might jump to the conclusion that it is sufficient to think of every point of a straight line as being paired with a rational number. But if we do hold this view for the moment, consider what will happen if we draw an arc with center at the point labeled 0 (zero) and with radius equal to the diagonal of a rectangle of length 2 and width 1. The arc will cross the straight line (Fig. 1) without having any point in common with the line. For the radius of the arc is the number whose square is 5, and it can be proved that there is no rational number, like \( \frac{p}{q} \), such that \( p^2 = 5 \). Consequently, if we think of every point of a straight line as being paired with the rational numbers, we must admit that these points, although ever so close together, are not so close but that the arc can sneak across the line without having a definite point of intersection. That is, the line so considered is not continuous.

Naturally we do not want to leave the matter like this. So to every such gap, or separation, in the rational numbers we assign a new sort of number, called an irrational number, and announce that we have decided to consider every point of a straight line as being paired either with a rational or with an irrational number. This makes the line continuous. The rational and irrational numbers together constitute the real numbers, and because of this connection with the points of a line the system of real numbers is said to be continuous.

The definition of every irrational number like \( \sqrt{5} \), \( -\sqrt{2} \), and \( \pi \) in terms of a separation of the rational
numbers seems very strange when we first encounter it. But if we must choose between a number system that fits "porous" lines only, and a strange new idea that enlarges our number system so that it fits continuous lines also, we naturally prefer the latter.

It is clear from the foregoing that the very first postulate of this geometry, the Principle of Line Measure, with its assumption that "the points on any straight line can be numbered so that number differences measure distances," implies that the number system to be used in this geometry is adequate to measure the distance between any two points whatsoever on the line. It implies, in other words, that this geometry is based upon the system of real numbers. This means that all the theorems derived from the five basic principles of this geometry are as true for irrational as for rational cases. And it means also, as will appear below, that we can think of a point as varying continuously along a straight line or along a curved line, and can think also of the continuous variation of an angle.

When a ball is thrown 58.2 feet by a baseball player, its distance from the player varies continuously from 0 to 58.2. In general, a quantity is said to vary continuously from a to b when it takes on all the values from a to b in order.
CONTINUOUS VARIATION OF A POINT IN A PLANE

A point is said to vary continuously in a plane if its coordinates \((x, y)\) with respect to a given pair of axes vary continuously. For example, \(P\) is said to vary continuously from \(A\) to \(B\) (Fig. 3) if \(y\) varies continuously from 2 to 7 as \(x\) varies continuously from -3 to 10.

\[
\begin{array}{c}
A: (-3, 2) \\
B: (10, 7) \\
\end{array}
\]

Fig. 3

CONTINUOUS VARIATION OF AN ANGLE

Strictly our statement of Principle 3 ought to have included a supplementary statement to the effect that the numbers assigned to half-lines having the same endpoint vary continuously along any intersecting line, whether straight or curved. Let us assume that Principle 3 has been thus amended.

It follows that if we have three points \(A\), \(B\), and \(X\) (Fig. 4) such that \(A\) and \(B\) remain fixed while \(X\) varies continuously from \(C\) to \(D\), then angles \(ABX\), \(BXA\), and \(XAB\) will vary continuously also.

Theorem 28 on the next page shows how the idea of continuous variation helps us prove theorems concerning inequalities.
**Theorem 28.** If one side of a triangle is greater than a second side, the angle opposite the first side is greater than the angle opposite the second side.

**GIVEN:** Triangle $ABC$ in which $c > b$.

**TO PROVE:** \( \angle p > \angle q \).

**PROOF:** We shall use the indirect method, and shall assume that \( \angle q > \angle p \). If now $B$ varies continuously along the arc of a circle of radius $a$ so that $p$ varies continuously toward $180^\circ$, as shown in Fig. 5, then $q$ varies continuously toward $0^\circ$. But $q$ is assumed at the outset to be greater than $p$. Therefore $q$, as it varies toward $0^\circ$, must somewhere be equal to the corresponding value of $p$. But this would make $a = b$, which is contrary to the hypothesis. Therefore $q$ cannot be greater than $p$. Nor can $q$ be equal to $p$. Therefore $q$ must be less than $p$, which is what we set out to prove.

**Theorem 29.** If one angle of a triangle is greater than a second angle, the side opposite the first angle is greater than the side opposite the second angle.

**GIVEN:** Triangle $ABC$ in which $c \angle q$.

**TO PROVE:** $a > b$.

**PROOF:** Prove the theorem. Use the indirect method and \( \triangle ABC \).

**Theorem 30.** If two sides of one triangle are equal respectively to two sides of another triangle, but the
Theorem 31. If two sides of one triangle are equal respectively to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

GIVEN: Triangles $ABC$ and $ABC'$ in which $BC=BC'$ and $Lq>Lq'$.

TO PROVE: $b>b'$.

PROOF: We shall use the indirect method and shall assume that $b'>b$. If $C'$ varies continuously along the arc of a circle of radius $a$, as shown in Fig. 7, $q'$ varies continuously from $q$ to $0$, and $b'$ varies continuously from $b$ to $a-c$. Since $b'$ has been assumed to be greater than $b$, $a-c$ must be greater than $b$. But this would mean that $a>b+c$, which is impossible; for one side of a triangle must always be less than the sum of the other two sides (Corollary 12c). Therefore $b'$ cannot be greater than $b$. Neither can it be equal to $b$. Why? Therefore $b'$ must be less than $b$.

Theorem 31. If two sides of one triangle are equal respectively to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

GIVEN: Triangles $ABC$ and $ABC'$ in which $BC=BC'$ and $AC>AC'$.

TO PROVE: $Lq>Lq'$.

PROOF: Prove the theorem. Use the indirect method and Theorem 30.
EXERCISES

*1. Prove by means of one of the foregoing theorems that if in the same circle, or in equal circles, two minor arcs are unequal, the greater arc has the greater chord. See Fig. 9.

*2. Prove that if in the same circle, or in equal circles, two chords are unequal, the greater chord has the greater minor arc. It is understood that neither of these chords is a diameter.

*3. Prove that if in the same circle, or in equal circles, two chords are unequal, the shorter is at the greater distance from the center. *Suggestion: The respective distances of the two chords from the center of the circle in Fig. 10 are $MO$ and $NO$. Draw $MN$. We can show $MO > NO$ if we can show $Ly > Lx$. How can we do this? Consider triangle $MBN$.

*4. Prove that if in the same circle, or in equal circles, two chords are unequally distant from the center, the one at the greater distance from the center is the shorter.

Exercises 3 and 4 are more easily proved directly by means of Principle 12, as in Exercises 4 and 6, page 137.

5. Prove that the sum of the four sides of any quadrilateral is greater than the sum of the diagonals.

CONTINUOUS VARIATION OF DIRECTED ANGLES

On page 209 we found that it is possible to make a distinction with respect to sign between the lengths of directed arcs. We now have a use for this idea. See Fig. 11 on the next page. We shall call directed arc $CA$...
positive when its directed central angle is positive, that is, counter-clockwise; and we shall call directed arc \( \overrightarrow{CA} \) negative when its directed central angle is negative, that is, clockwise. We shall make a similar distinction between the positive and negative directed arcs \( \overrightarrow{AC} \).

In other words, the sign of a directed arc will be the same as the sign of the corresponding directed angle at the center of the circle. If you need to refresh your understanding of this idea of directed angles, turn back to the explanation on page 49.

We can use the notations \( LCOA^+, LCOA^-, LAOC^+, \) and \( LAOC^- \) to distinguish the four directed angles formed by the half-lines \( OC \) and \( OA \). Similarly, we can use the notations \( \overrightarrow{CA}^+ \) and \( \overrightarrow{CA}^- \) to distinguish the respective lengths of the positive and negative directed arcs \( \overrightarrow{CA} \), and \( \overrightarrow{AC}^+ \) and \( \overrightarrow{AC}^- \) to distinguish the lengths of the other pair of directed arcs. Upon occasion, where the length of the directed arc is not in question and there is no risk of confusion, we may for convenience replace the notation for a particular directed central angle, for example \( LCOA^+ \), by the somewhat simpler notation \( \overrightarrow{CA}^+ \) for its corresponding directed arc. But whenever we make this shift in notation we must be sure to state clearly just what we are doing.

We can now restate Theorem 22 as follows: An inscribed directed angle is equal to half the directed central angle having the same arc. This assumes that neither directed angle is greater than \( 360^\circ \). Restated in this form the theorem has little practical value, but the idea is fascinating when it is applied to the theorems in Exercises 5-10, pages 147-148, for it enables us to summarize all of these theorems in one general statement, as in the first paragraph on the next page.
If two straight lines $AB$ and $CD$ through a given point $P$ cut a given circle in the points $A$, $B$ and $C$, $D$ respectively (Fig. 12), then the directed angle from the first line to the second line is equal to half the sum of the directed central angles having the directed arcs $AC$ and $BD$. This is true whether $P$ be inside the circle, on the circle, or outside the circle. That is, regardless of the position of $P$, directed $LBPD = \frac{1}{2} (\text{directed } LAOC + \text{directed } LBOD)$. This will be apparent after you have studied Fig. 13 on page 239 with the help of Exercises 1 to 11 below.

**EXERCISES**

The best way to appreciate the significance of continuous variation is to study the following exercises and satisfy yourself that in each case the printed statement is correct. You are not expected to formulate a detailed proof for each exercise. It is more important that you observe the gradual change in the geometry as you pass from Ex. 3 to Ex. 11, and that you appreciate the permanence of form in the algebra that expresses this continuous variation in the geometry.

In each exercise below you are to consider the three points $A$, $B$, and $D$ as fixed. You are then asked to observe that the continuous variation of a fourth point produces a continuous variation in an angle or in a fifth point, and to note how one continuous variation affects the other.

1. Show that in Fig. 12 the positive directed angle $x$ varies continuously from almost $0^\circ$ to almost $180^\circ$ as $P$ varies continuously along the line $AD$ (extended).
2. In Fig. 12 on page 236 consider points A, B, and D as fixed and show by a series of sketches that \( P \) varies continuously along \( AB \) (extended) as \( C \) varies positively, that is, counter-clockwise around the circle.

In Exercises 3-11, \( C \) varies continuously in counter-clockwise direction around the circle. This produces a continuous variation in the positive directed angle \( x \) from the line \( AB \) (extended when necessary) to the line \( CD \) (extended when necessary). In studying this variation we shall denote the measure of the positive directed angle \( x \) by the symbol \( X^+ \), and shall use the symbols \( BD^+ \) and \( AC^+ \) to denote the measures of the positive directed central angles that correspond to the positive directed arcs \( BD \) and \( AC \) respectively.

3. In circle \( a \), Fig. 13, page 239, use the fact that \( \angle LBPD \) is equal to \( \frac{1}{2}(\angle BOD - \angle COA) \), which you proved in Ex. 6, page 148, to show that the directed angle \( x^+ \) is equal to half the sum of the directed central angles having the directed arcs \( BD \) and \( AC \). That is, show that

\[ x^+ = \frac{1}{2}(BD^+ + AC^+). \]

4. In circle \( b \), Fig. 13, show that \( x^+ = \frac{1}{2}(BD^+ + AC^+). \)

5. In circle \( c \), Fig. 13, show that \( x^+ = \frac{1}{2}(BD^+ + AC^+). \)

6. In circle \( d \), Fig. 13, show that \( x^+ = \frac{1}{2}(BD^+ + AC^+). \)

7. In circle \( e \), Fig. 13, show that

\[ x^+ = \frac{1}{2}(360) - Y^+, \]

and \( Y^+ = \frac{1}{2}(DA^+); \)

whence \( x^+ = \frac{1}{2}(360 - DA^+) = \frac{1}{2}(BD^+ + AC^+). \)

8. In circle \( f \), Fig. 13, show that

\[ x^+ = \frac{1}{2}(360) - Y^+, \]

and \( Y^+ = \frac{1}{2}(DA^+ + BC^+); \)

whence \( x^+ = \frac{1}{2}(360 - DA^+ + BC^+) = \frac{1}{2}(AD^+ + BC^+) = \frac{1}{2}(BD^+ + AC^+). \)

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9. In circle $g$, Fig. 13, show that $x = \frac{1}{2}(BD + AC)$.

10. In circle $h$, Fig. 13, show that $x = \frac{1}{2}(BD + AC)$.

11. In circle $i$, Fig. 13, CD is parallel to $AB$. Show that as C varies continuously from the position shown in circle $h$, Fig. 13, to the positions shown in circles $j$ and $k$, $x$ is always equal to $\frac{1}{2}(BD + AC)$.

A single general statement serves also to summarize the theorems of Exercises 35-38, page 151, as was indicated in Ex. 46, page 152: If a straight line through a given point $P$ cuts a given circle in the points $A$ and $B$, then the product $PAXPB$ is the same for every possible position of the straight line. This is true whether the given point $P$ is inside the circle, on the circle, or outside the circle. Now let us consider how this product varies as the position of the given point $P$ varies with respect to the circle. We shall use Fig. 13 for this purpose and shall employ directed distances, as described on page 42. The different diagrams in Fig. 13 show different positions of the given point $P$; and $PAB$ or $APB$ or $ABP$ represents the random line that cuts the circle in $A$ and $B$. This random line through $A$ and $B$ is kept the same in all the diagrams in order to systematize the variation of the point $P$.

12. In circle $a$, Fig. 13, $P$ varies continuously toward $B$ along $AB$ extended, but so that the three points are always in the order $P, A, B$. What can you say about the sign and size of $PA$, $PB$, and $PAXPB$? How does the product $PAXPB$ vary?

13. In circle $b$, Fig. 13, what does the product $PAXPB$ equal?

14. In circle $c$, Fig. 13, $P$ varies continuously along $AB$ from $A$ to $B$. What can you say about the sign of $PA$, $PB$, and $PAXPB$? 238
Fig. 13
15. In circle $e$, Fig. 13, what does the product $PA \times PB$ equal?

16. In circle $t$, Fig. 13, $P$ varies continuously from $B$ along $AB$ extended, but so that the three points are always in the order $A, B, P$. What can you say about the sign and size of $PA, PB,$ and $PA \times PB$?

17. Show that as $P$ varies continuously along the fixed line $AB$, the product $PA \times PB$ decreases from indefinitely large positive values through zero to negative values, and then increases through zero again to indefinitely large positive values.

18. We have just seen in Ex. 17 that for all positions of the point $P$ between $A$ and $B$ the product $PA \times PB$ is negative. Find that position of the point $P$ for which $PA \times PB$ has the largest negative value. Show that this value is $-\frac{(AB)^2}{4}$. 

*Suggestion: We can do this geometrically by thinking of the unsigned product $PA \times PB$, or $rs$, as representing the area of a rectangle of sides $r$ and $s$. Fig. 14 shows three such rectangles. We must then discover which of all possible rectangles of sides $r$ and $s$ will have the largest area. What do you know about the perimeters of these rectangles? For a further suggestion, see page 218.*

We can also use an algebraic method to discover when the product $AP \times PB$ has the largest positive value. For we can rewrite $AP \times PB$ in the form $\frac{\alpha \cdot \delta}{2} - PM \times \frac{\alpha \cdot \delta}{2} + PM \cdot \delta$ and need only discover the condition under which this product is as large as possible.
The large ocean liner pictured above is bound from Southampton to New York. When about half a day's run from New York the captain wished to determine his position as accurately as possible. By means of his radio direction finder he discovered that at 9:14 P.M. his ship was somewhere on a line passing through Cape Cod Light at an angle of 48° west of true north, and also somewhere on a line passing through Nantucket Lightship at an angle of 82° west of true north. The approximate position of the ship was at the intersection of these two lines, as shown in the diagram below. Each of these lines, when regarded in this manner, is called a "locus," as you will learn in Chapter 9.
Loci

This chapter deals with geometric facts already familiar to you. The chief novelty is in the point of view; but even that is partly familiar from some of the situations you have met in the preceding chapters. Perhaps the only aspect of this subject that will seem entirely strange is the name that is used for it. We call it "locus," which is the label that has become attached to this subject through the ages. The plural of "locus" is "loci." Do not be awed by the word "locus." Instead, use it in simple situations until you become familiar with it and know what it means.

A geometric locus is a figure which includes all the points that satisfy a given condition (or conditions), and no other points. Sometimes it is convenient to regard a locus as the path of a point moving according to a fixed law,* but most of the locus situations you will meet do not

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*As already indicated on page 60, our geometry is logically independent of the idea of motion. If we wish to entertain the idea of motion, however, we can develop it from the idea of number, which is fundamental in our geometry. For we can define the motion of a point $P$ in a plane as the continuous variation of its coordinates $(x, y)$. If we wish to entertain the idea of congruent geometric figures (see pages 59 and 60), where one figure is moved and fitted on the other, we can define this fitting or congruence by saying that to every point $(x, y)$ in the first figure there corresponds a point $(x', y')$ in the second figure, such that all corresponding distances are equal.
require the idea of motion. The word “locus,” like the word “location,” comes from the Latin. Use the dictionary to find the original Latin meaning of the word.

Locus A. The locus of all the points in a plane at a given distance from a given fixed point is a circle whose center is the given point and whose radius is the given distance. See Fig. 1.

What is the locus of all the points in space which are 3 inches from a fixed point O?

Locus B. The locus of all the points in a plane at a given distance from a given endless line is a pair of lines, one on each side of the given line, parallel to it and at the given distance from it. See Fig. 2.

What is the locus of all the points in space which are 2 inches from a given line?

When a locus is regarded as the path of a moving point, it is not easy to describe it using the easily understood phrase “The locus of all points that . . . .” Instead, we usually describe such loci by saying, “The locus of a point that . . . .” You may find it helpful in such cases to think of this as meaning “The locus of all positions of a point that . . . .”

What is the locus of the center of a circle which rolls around outside a fixed circle, always just touching it? See Fig. 3. What is the locus if the rolling circle is small enough to roll around inside the fixed circle?

If two circles have equal radii, and if one of these circles rolls around outside the other, what is the locus of the center of the rolling circle?
Describe the locus of the center of a circle which rolls around outside a square, always just touching it.

A locus often includes isolated points as well as straight lines and curves, and sometimes only points, as in Ex. 3 below. In such cases the locus, although conforming to the original definition in that it includes all the points that satisfy a given condition (or conditions), cannot be considered as the path of a moving point.

**EXERCISES**

You need not prove the correctness of your answers for the following exercises. The chief purpose of these exercises is to help you become familiar with various situations concerning loci. Later in the chapter you will be asked to prove exercises concerning loci.

1. What is the locus of all the points in a plane which are 5 inches from a fixed point 0 in the plane?

2. What is the locus of all the points in a plane which are 4 inches from an endless fixed line lying in the plane? What is the locus when the fixed line is perpendicular to the given plane?

3. What is the locus of all the points in a plane which are 5 inches from a fixed point 0 in the plane and at the same time 4 inches distant from a fixed line \( AB \) that is in the plane and that passes through O? Make a careful drawing and indicate the locus accurately. **Suggestion:** What is the locus of all points 5 inches from O? What is the locus of all points 4 inches from \( AB \)? What points are on both loci, thereby satisfying both conditions?

4. What would be the locus in Ex. 3 if \( AB \), instead of passing through O, were 1 inch from O? Make a careful drawing and indicate the locus accurately.
5. What is the locus of all the points 2 inches from each of two given parallel lines that are 4 inches apart?

6. What is the locus of all the points 3 inches from each of two given parallel lines that are 4 inches apart?

In each of the following exercises assume that the desired locus is confined to a plane.

In plotting a locus enough points should be determined to indicate the nature and position of the locus. The locus should then be drawn so as to include all these points. It is helpful to consider extreme positions for the points and lines involved. Sometimes these limiting cases produce a marked alteration in the locus, or even cause it to vanish.

When one side of a triangle is singled out for special reference, it is called the base. That vertex which lies opposite the base is called the vertex.

7. Plot the locus of the vertex of a triangle whose base is a fixed line segment and whose altitude is constant.

8. Circles are drawn so as to have a fixed line segment as their common chord. Plot the locus of the mid-points of the lines which join the centers of the circles to one extremity of the chord.

9. A point moves so that its distance from a given line is always one-half its distance from a fixed point in the line. Plot its locus.

10. Plot the locus of the intersection of tangents to a circle at the extremities of a variable chord which passes through a given point in the circle. See Fig. 4. Suggestion: In this, as in all other locus exercises, consider the limiting positions of the chord. What happens to the locus if \( P \) is at the center of the circle?
11. A square rolls along a straight line without slipping. Plot the locus of one corner of the square as the square makes one complete revolution.

12. A thin straight stick one foot long floats on water in a circular basin 20 inches in diameter. Sketch the surface of the water, and on this sketch shade that portion of the surface that is accessible to the mid-point of the stick. Describe the boundary of this portion accurately.

13. A point moves so that its distances from a fixed point and a fixed line are equal. Plot the locus.

14. A point moves so as always to be half as far from a fixed line \( l \) as from a fixed point not on \( l \). Plot the locus.

15. Plot the locus of all points the sum of whose distances from two fixed points 3 inches apart is always the same, say 5 inches.

16. A straight line 3 inches long moves so that each end is always on the boundary of a 4-inch square. Plot the locus of the middle point of the moving line.

The definition of locus on page 242 contains the significant expression “all the points that satisfy a given condition, and no other points.” As already indicated on page 88, the wording “all the points . . . and no other points” is equivalent to an “If and only if . . . ” phrasing, that is, to a proposition and its converse. Let us examine this more closely.

The definition of locus says two things:

1. If a point satisfies the given conditions, it lies on (in other words, is a point of) the locus.
2. If a point does not satisfy the given conditions, it does not lie on the locus.

We say that each of these propositions is the opposite of the other. We must prove both of them whenever
we wish to prove that a supposed locus is the whole locus and nothing but the locus.

But we can show that whenever the opposite of a proposition is true, the converse of the proposition is true also. In this case, if we call proposition 1 on page 246 the "direct" proposition and proposition 2 its "opposite," then the converse of the direct proposition would be:

3. If a point lies on the locus, it satisfies the given conditions.

We can prove this at once by the indirect method. For suppose that the conclusion of 3 is not true and that the point does not satisfy the given conditions; then by 2 it does not lie on the locus, and this contradicts the hypothesis of 3. Therefore, when 2 is true, 3 is true also.

Instead, therefore, of establishing a locus rigorously by proving a proposition and its opposite, we may prove the proposition and its converse; and usually this will be the easier thing to do.

You can now show that if 3 is true, 2 is true also. In fact, every proposition in geometry is related to three others as indicated in the accompanying table.

1. Proposition
   If $A$ is true, then $B$ is true.

2. Opposite
   If $A$ is not true, then $B$ is not true.

3. Converse
   If $B$ is true, then $A$ is true.

4. Opposite Converse
   If $B$ is not true, then $A$ is not true.

The diagonal lines indicate that 1 and 4 amount to the same thing: if either is true, both are true. Similarly, if either 2 or 3 is true, both are true. In other words, if a proposition is true, its opposite converse is true; for 3 is just as much the opposite converse of 2 as 4 is of 1. In every case this can be proved by the indirect method, following the pattern of the proof given above.
The method of establishing a locus rigorously by proving a proposition and its converse will become clear to you as you study the seven standard locus theorems that follow. No other theorems in this book require this method of proof. You will find the seven locus theorems very useful later on when you are asked to solve exercises involving loci.

**THE SEVEN STANDARD LOCUS THEOREMS**

*Locus Theorem 1.* The locus of all the points in a plane at a given distance from a given fixed point is a circle whose center is the given point and whose radius is the given distance.

This follows immediately from our definition of “circle” on page 133 in Chapter 5.

*Locus Theorem 2.* The locus of all the points in a plane at a given distance from a given endless line is a pair of lines, one on each side of the given line, parallel to it and at the given distance from it.

![Diagram](image)

Fig. 5

**GIVEN:** Line \( l \) and random points \( P, Q, R, P', Q', R' \) on either side of \( l \) and at a distance \( d \) from it.

**TO FIND:** The locus of all such points.
PROOF: From all these points choose any two that are on the same side of \( l \), such as \( P \) and \( Q \), and draw \( PQ \). See Fig. 5.

Since the distance from a point to a line is measured along the perpendicular from the point to the line, \( PD \) and \( QE \) are both perpendicular to \( l \) and therefore parallel. Why? But \( PD = QE \) (Given). Therefore \( PQED \) is a parallelogram, and \( PQ \) is parallel to line \( l \) and at a distance \( d \) from it.

Had we chosen \( R \) instead of \( Q \), we should have \( PR \) parallel to line \( l \). Therefore \( PR \) and all similar lines coincide with \( PQ \), and \( R \) lies on \( PQ \). Similarly \( P' \), \( Q' \), and \( R' \) lie on a line parallel to \( l \) and at a distance \( d \) from \( l \).

We have proved that every point satisfying the given conditions lies on the supposed locus. We must now prove the opposite, or else the converse, of this. The converse is easier; so we shall show that every point \( S \) on the supposed locus must satisfy the given condition that it be at a distance \( d \) from \( l \).

\( PS \) is given parallel to \( l \), and \( PD \) is perpendicular to \( l \) and equal to \( d \). Draw \( SG \) perpendicular to \( l \). Then \( SG \) will be parallel to \( PD \) (Why?) and equal to \( d \) (Why?).

**Locus Theorem 3.** The locus of all points equidistant from two given parallel lines is a line parallel to the given lines and midway between them.

**GIVEN:** The parallel lines \( l \) and \( m \) at a distance \( d \) from each other. See Fig. 6.

**TO FIND:** The locus of all points equidistant from \( l \) and \( m \) and lying in their plane.

**PROOF:** By Locus The-
Locus Theorem 2. All points at a distance $\frac{d}{2}$ from $l$ lie on two lines $p$ and $q$ parallel to $l$ and at a distance $\frac{d}{2}$ from it. And similarly $p$ and $r$ are the locus of all points at a distance $\frac{d}{2}$ from $m$. Line $p$ is common to both loci and so contains all points at a distance $\frac{d}{2}$ from both $l$ and $m$.

Conversely, by Locus Theorem 2, every point on $p$ is at a distance $\frac{d}{2}$ from both $l$ and $m$.

What is the locus if the points are not confined to the plane of $l$ and $m$?

Locus Theorem 4. The locus of points equidistant from two given points is the perpendicular bisector of the line segment joining these points.

This is merely restating Principle 10 in terms of locus.

**GIVEN:** The points $A$ and $B$ and any point $P$ such that $AP = BP$.

**TO FIND:** The locus of all such points $P$.

**PROOF:** Draw $PM$ from $P$ to the mid-point $M$ of $AB$. See Fig. 7. In triangles $AMP$ and $BMP$, $LAMP = LBMP$. Why? Therefore each of these angles is a right angle (Why?), and $PM$ is the perpendicular bisector of $AB$. This proves that every point that is equidistant from $A$ and $B$ lies on the perpendicular bisector of $AB$.

We must now prove the opposite, or else the converse, of this proposition. The converse is easier; so we shall prove that any point $Q$ on the perpendicular bisector of $AB$ is equidistant from $A$ and $B$. This follows imme-
Fig. 8

Fig. 9

GIVEN: Lines \( l \) and \( m \) intersecting at \( O \), and any point \( P, Q, R, \) or \( S \) equidistant from \( l \) and \( m \). See Fig. 9.
TO FIND: The locus of all such points.

PROOF: From $P$ drop perpendiculars $PA$ and $PB$ to the given lines $l$ and $m$; draw $PO$. In the right triangles $AOP$ and $BOP$, $AP = BP$ (Given). Therefore $\angle AOP = \angle BOP$ (by Corollary 12b). This means that $PO$- and similarly also $QO$, $RO$, and $SO$-bisects the angle formed by $l$ and $m$. It follows that all points equidistant from $l$ and $m$ lie on these bisectors.

Conversely, every point $T$ on a bisector is equidistant from $l$ and $m$. For if perpendiculars be dropped from $T$ to $l$ and $m$, application of Case 2 of Similarity shows that these perpendiculars are equal.

What is the locus if the points are not confined to the plane of the given lines?

What is the locus of points equidistant from two given planes?

**Locus Theorem 6.** The locus of the vertex of a right triangle having a given hypotenuse as base is the circle whose diameter is the given hypotenuse. The locus does not include the end-points of the hypotenuse.

**GIVEN:** Points $A$ and $B$ and a random point $V$ such that $VA$ is perpendicular to $VB$. See Fig. 10.

**TO FIND:** The locus of $V$.

**PROOF:** Instead of proving the direct proposition that every point $V$ lies on the circle, we shall find it easier to prove its equivalent, the opposite converse: For any point $P$ not on the circle the angle $APB$ is not a right angle. Prove this. See Exercises 5 and 6, pages 147 and 148.

The converse, that every point on the circle, except $A$ and $B$, is the vertex of a right triangle having $AB$ for hypotenuse, follows at once from Corollary 22c.
Referring to the table on page 247, we see that we have proved this locus theorem by proving 4 and 3 instead of 1 and 3.

**Locus Theorem 7.** The locus of the vertex of a triangle having a given base \( AB \) and a given angle at the vertex is a pair of equal circular arcs \( AB \), symmetric with respect to \( AB \), but not including \( A \) and \( B \).

This can be proved in the same way as Locus Theorem 6 by referring to Exercises 5 and 6, pages 147 and 148, and to Corollary 22b. See Fig. 11.

The actual construction of the locus is done as follows. Let \( AB \) (Fig. 12) be the given base of the triangle and \( \angle V \) the given angle. At \( B \) construct \( \angle ABT \) equal to \( LV \). Construct the perpendicular to \( TB \) at \( B \). Also construct the perpendicular bisector of \( AB \). These two lines will meet at \( O \), the center of one of the desired circles.

For every point on the perpendicular bisector of \( AB \) is the center of a circle through \( A \) and \( B \), and \( O \) is the center of a circle which is tangent to \( TB \) at \( B \). Why?

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Therefore $LABT$ is equal to half the central angle having the arc $AB$, and $LAVB$ is also. Arc $AVB$ is half the required locus. The center $O'$ of the other arc can be found by laying off on the perpendicular bisector the length $MO'$ equal to $MO$.

**EXERCISES**

In each exercise below first state precisely what the locus consists of, unless it is already stated in the exercise, and then prove that this is right. You may use any of the seven locus theorems in your proof. Usually this will save you the trouble of going through the details of proving a theorem and its converse in order to prove the locus correct. Exercises 1-48 concern plane geometry only.

1. Find the locus of the vertices of isosceles triangles having a common base.

2. Find the locus of the mid-points of straight line segments joining two parallel lines. See Fig. 13.

*3. Find the locus of the mid-points of lines drawn from a given point to a given line.

4. The base $BC$ of a triangle $ABC$ is a fixed line segment, and the vertex $A$ moves so that the area of the triangle is always the same. Find the locus of $A$.

5. $AB$ is a fixed straight line segment. At each point of $AB$ a perpendicular is erected equal in length to the distance of the point from $A$. What is the locus of the upper extremities of these perpendiculars?

6. What is the locus of the centers of circles which are tangent to the line $XY$ at the point $P$?

7. Find the locus of the centers of all circles having a given radius and passing through a given point.
8. Find the locus of the centers of all the circles which pass through two fixed points.

*9. Find the locus of the mid-points of all chords of a given length drawn in a given circle.

10. Find the locus of the centers of circles which cut equal chords from two fixed parallel lines.

*11. Prove that the line joining the center of a circle and the point of intersection of two tangents is the perpendicular bisector of the chord joining the points of contact.

*12. Prove that the line drawn from the mid-point of a chord to the mid-point of its minor (or major) arc is perpendicular to the chord.

*13. Prove that the perpendicular bisector of a chord passes through the center of the circle and through the mid-points of both arcs.

*14. Prove that if two circles intersect, the line of centers is the perpendicular bisector of their common chord.

*15. Prove that the line which passes through the mid-points of two parallel chords passes through the center of the circle.

*16. A system of parallel chords is drawn in a circle. Find the locus of the mid-points of these chords.

17. All but one small section of a barrel head is lost (Fig. 14). Make a drawing to show how you would find the radius of the original barrel head.

*18. Prove that the perpendicular bisectors of the sides of a triangle are concurrent in a point that is equidistant from the vertices. *Suggestion:* Show first that the perpendicular bisectors of two sides must meet in a point which is equidistant from all three vertices. This point is called the circumcenter of the triangle because it is the center of the circle circumscribed about the triangle.
19. Prove that the bisectors of the angles of a triangle are concurrent in a point that is equidistant from the sides of the triangle. This point is called the *incenter* of the triangle because it is the center of the inscribed circle.

20. Prove that through three points not in a straight line one circle, and only one, can be drawn.

21. Prove that a circle cannot ordinarily be drawn through four points taken at random. That is, show that the reasoning employed in Ex. 20 will not hold in general for four points.

22. Circles are drawn so that they are always wholly within a given triangle and tangent to two of its sides. Find the locus of the centers of these circles.

23. Find the locus of points whose distances from two fixed parallel lines are in a given ratio.

24. Construct the tangent at a given point \( P \) of a given circle, the center of the circle not being accessible. See Fig. 15. Prove that your construction is correct.

25. Find the locus of all points from which tangents of a given length can be drawn to a given circle.

26. What is the locus of the mid-points of chords of a given circle formed by secants drawn from a given external point?

27. Find the locus of points of contact of tangents drawn from a given external point to concentric circles.

28. A 60° angle moves so that both of its sides touch a fixed circle of radius 5 feet. Find the locus of the vertex.

29. Triangles are constructed on a given line segment as base. What is the locus of the foot of the perpendicular drawn from the mid-point of the base to either of the other two sides?
30. The hypotenuse of a right triangle is a given line segment. What is the locus of the mid-point of either of the other sides?

*31. Find the locus of the mid-points of chords drawn from a given point on a circle.

*32. Find the locus of the mid-points of chords drawn through a given point within a given circle.

*33. A straight rod moves so that its ends always touch two fixed rods that are perpendicular to each other. Find the locus of its mid-point. See Fig. 16.

*34. Prove that if two opposite angles of a quadrilateral are supplementary (that is, add up to 180°) the quadrilateral can be inscribed in a circle. This is the converse of Ex. 1, page 147. Use the indirect method and Exercises 5 and 6, pages 147 and 148.

35. A and B are two fixed points on a given circle, and CD is a variable diameter. Find the locus of the intersection of CA and DB.

*36. Upon a fixed line segment AB (Fig. 17) a segment of a circle containing 240° is constructed (that is, the arc of the segment is \( \frac{240}{360} \) of the circumference), and in the segment any chord CD having an arc of 60° is drawn. Find the locus

(a) of the point of intersection of AC and BD;

(b) of the point of intersection of AD and BC.

37. Triangles are constructed on a fixed line segment AB as base and with a given angle at the vertex. Find the locus of the intersection of the perpendiculars from A and B to the opposite sides of the triangle.
38. Lines are drawn parallel to the base of a triangle and are terminated by the other two sides. What is the locus of their mid-points?

39. Two railroad lines cross at an angle of 114°. See Fig. 18. How would you locate the center of a circle with a 100-foot radius which would join one line with the other? Suggestion: The circle must be tangent to the two lines.

40. Two circles intersect at the points $A$ and $B$. Through $A$ a variable secant is drawn, cutting the circles at $C$ and $D$. Prove that the angle $DBC$ is constant.

41. In Fig. 19 show how to find the point $P$ on the straight line $MR$ such that the broken line $APB$ is the equal-angle route from $A$ to $B$. We can think of $MR$ as a mirror or as the cushion of a billiard table.

42. In Fig. 20 show how to find the point $Q$ on the straight line $MR$ such that the broken line $AQB$ is the shortest route from $A$ to $B$ that includes a point of $MR$.

Exercises 41 and 42 show that the equal-angle route from $A$ to $B$ and the shortest route from $A$ to $B$ are the
same. Physicists have observed that a ray of light, in going from one point to another, goes always in the shortest possible time.

*43. Prove that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent. See Fig. 21. 

**Suggestion:** Through each vertex draw a line parallel to the opposite side of the triangle. Show that these three lines form a triangle the perpendicular bisectors of whose sides are the given perpendiculars from the vertices of the given triangle. See Ex. 18, page 255. The perpendiculars from the vertices of triangle $ABC$ to the opposite sides intersect at $O$, which is called the orthocenter of the triangle.

The line joining a vertex of a triangle to the mid-point of the opposite side is called a MEDIAN of the triangle.

*44. Prove that the medians of a triangle are concurrent in a point that is two-thirds of the distance from each vertex to the middle of the opposite side. **Suggestion:** Two medians, such as $AL$ and $BM$ in Fig. 22, meet at $G$. If $X$ is the mid-point of $AG$, and $Y$ the mid-point of $BG$, show that $XY$ and $ML$ are parallel to $AB$ and equal to $\frac{1}{2}AB$. Then show that $AG = \frac{2}{3}AL$, and that $BG = \frac{2}{3}BM$. That is, any median meets any other median at a point which is two-thirds of the distance from a vertex to the mid-point of the opposite side. This point is called the centroid, or center of gravity, of the triangle.

Exercises 18, 19, 43, and 44 form an interesting group of theorems concerning concurrent lines in a triangle.
45. Prove that if each of three circles is tangent externally to the other two, the common internal tangents are concurrent. *Suggestion:* Consider the relation of the point of intersection of each pair of common internal tangents to the sides of the triangle formed by joining the centers of the three circles.

46. Given two intersecting lines $AB$ and $AC$, what is the locus of all points that are twice as far from $AB$ as from $AC$? *Suggestion:* Prove first that if $P$ is one such point, every point on $AP$ is twice as far from $AB$ as from $AC$. Then prove conversely that if $P$ is one such point, another point $P'$ that is twice as far from $AB$ as from $AC$ must lie on $AP$ (extended).

*47. Prove that in triangle $APB$ (Fig. 23) the bisectors of the interior and exterior angles at $P$ divide $AB$ internally and externally in the same ratio, $\frac{m}{n}$. See Exercises 25 and 26, page 116.

Fig. 23

48. Find the locus of points whose distances from $A$ and $B$ are in the constant ratio $\frac{m}{n}$. Consider triangle $PQR$ (Fig. 23) and Ex. 27, page 117.

49. Reconsider Ex. 23 on page 256 as a locus of points not restricted to the plane of the given parallel lines; that is, find the locus of all points in space the distances of which from two fixed parallel lines are in a given ratio.
50. Point $P$ (Fig. 24) is 4 inches from a plane. Find the locus of all points in the plane which are 5 inches from $P$.

51. What is the nature of the intersection of a plane and a sphere? Prove it. See Ex. 50.

52. What is the nature of the intersection of two spheres?

**POWER OF A POINT**

*The power of a point $P$ with respect to a circle is the product of the two distances $PA$ times $PB$ from the point to the circle measured along a random secant.*

*If the point is inside the circle, its power is negative; if it is on the circle, its power is zero; and if it is outside the circle, its power is positive.* The power $PA \times PB$ (Fig. 25) is equal to $(PT)^2$ when $P$ is outside the circle and increases as $P$ recedes from the center. For a given point inside the circle the product $PA \times PB$ is constant no matter what secant through $P$ is chosen. In particular we may choose the diameter through $P$. We have already seen (Ex. 18, page 240) that the product $PA' \times PB'$ is numerically largest when $P$ is at the mid-point of $A'B'$. Consequently the largest negative power that a point can have with respect to a circle is $-r^2$; this is the power of the point 0 at the center of the circle.
EXERCISES

1. Prove that if two circles intersect, every point on the common chord, and on the common chord extended, has the same power with respect to both circles. See Fig. 26.

2. Prove that if two circles are externally tangent, every point on the common internal tangent has the same power with respect to both circles.

3. State and prove the corresponding theorem for two circles which are internally tangent.

RADICAL AXIS

All points whose powers with respect to two non-intersecting circles are equal lie on a straight line perpendicular to the line of centers of the given circles. This perpendicular is called the RADICAL AXIS of the circles.

We shall prove this by showing that whatever point of equal power is chosen, the perpendicular drawn through it to the line of centers always meets the latter at the same point. See Fig. 27.

For \((OD)^2 = (PO)^2 - (PD)^2 = (OT)^2 + (TP)^2 - (PD)^2\), and \((DO')^2 = (PO')^2 - (PD)^2 = (O'T')^2 + (TP)^2 - (PD)^2\);

whence \((OD)^2 - (DO')^2 = (OT)^2 - (O'T')^2\). Therefore the product \((OD-DO')(OD+DO')\) is constant; and since \((OD+DO')\) is constant, \((OD-DO')\) must be constant also.
\[ OD + DO' = k_1, \]
\[ OD - DO' = k_2, \]
\[ 2(OD) = k_1 + k_2, \]
and \[ OD = \frac{k_1 + k_2}{2}, \] another constant.

That is, \( D \), the foot of the perpendicular from \( P \) to \( 00' \), is the same for all possible positions of the point \( P \). Therefore all the points \( P \) must lie on the perpendicular to \( 00' \) at \( D \), the radical axis of the two circles.

**EXERCISES**

1. Show that the foregoing proof can be applied to two intersecting circles, and that the line \( PD \) must be the common chord (extended) of the two circles.

2. If now we agree to extend the meaning of "radical axis" to apply also to the common chord (extended) of two intersecting circles, what shall we mean by the radical axis of two circles that are externally tangent? Of two circles that are internally tangent?

3. What is the locus of all points having the same power with respect to two spheres?

**INVERSION**

Two points are said to be inverses of each other with respect to a circle when the product of their distances from the center is equal to the radius squared. In Fig. 28, \( OXOP' = r^2 \), \( P' \) is the inverse of \( P \), and \( P \) is the inverse of \( P' \). The circle is called the **cmCle of Inversion**: its center, the CENTER OF INVERSION.

The inverse of a point inside the circle is outside the circle; and conversely. The inverse of a point on the
circle is the point itself. To every point inside the circle, except the center, there corresponds one and only one point of the plane outside the circle. And for every point in the plane outside the circle, however small, there is a corresponding point inside the circle. Similarly, for all points in space outside a sphere, however small, there may be found corresponding points within the sphere.

The inverse of a line, whether straight or curved, is the locus of the inverse of every point of the line. For example, the inverse of a straight line through the center of inversion 0 is the line itself. You can see this by considering in turn the inverse of each point of the line.

What is the inverse of the circle of inversion?

The inverse of a circle with center at 0, other than the circle of inversion, is another circle with center at O. What is the relation between the radii of the three circles?

The inverse of a straight line through P' (Fig. 29) perpendicular to OP is the circle having OP as its diameter. For if Q' be any point on this perpendicular, its inverse Q will be such that

\[ OQ \cdot OQ' = r^2 = OP \cdot OP', \]

or

\[ OQ = OP' \]

Hence \( \angle OQP \) is a right angle (Why?), and so Q lies on the circle having OP as its diameter.

**EXERCISES**

1. Show that the foregoing proof holds when \( P' \) is on the circle of inversion, and also when \( P' \) is inside the circle. In general, the inverse of a straight line (not through 0) is a circle through the center of inversion.
2. What is the inverse of a circle which passes through the center of inversion?

**PROJECTION**

The feet of the perpendiculars dropped from every point of a given curve $AB$ (Fig. 30) to a line $l$ constitute the projection of $AB$ on $l$. In this sort of projection the auxiliary lines $AA'$, $PP'$, $BB'$ are all parallel.

What is the projection of a circle upon a line tangent to the circle?

What is the projection of the edge of a fifty-cent piece on a plane parallel to the plane of the coin? What is the projection if the plane is not parallel to the plane of the coin?

What is the projection of a sphere on a plane? Does this differ from the shadow which the sphere would cast on a sunlit plane?

**CENTRAL PROJECTION**

Another sort of projection, called CENTRAL PROJECTION (Fig. 31), is obtained by drawing half-lines from a fixed point $O$ through every point of $AB$ and determining where each such half-line intersects $l$. The intersection, $P'$, of $OP$ with $l$ is the projection of $P$ on $l$.

If the curve $AB$ is a semicircle, and if $l$ is parallel to
Fig. 32
The wisdom of the decisions reached in a town meeting, in a consultation between doctors, or in a trial by jury depends upon logical reasoning. Such reasoning in non-mathematical situations will be our goal in Chapter 10. Our geometry appears in this chapter as a concrete example of an abstract logical system. This geometry can serve, therefore, as an ideal pattern for all our logical reasoning.
CHAPTER 10

Reasoning. Abstract Logical Systems

IN THE OPENING PAGES of this book you were asked to decide two disputes from everyday life. In each of these you found it difficult to come to a clear-cut decision. It was apparent that, even though much of the world's thinking is concerned with everyday situations of this sort, you would find it difficult to learn the rules of logical argument merely by studying such situations. It was decided, therefore, to turn to abstract, impersonal reasoning in geometric situations in order to get practice in logical argument and in order to see what constitutes a coherent, logical system.

After studying four chapters of this book devoted to reasoning in important geometric situations, you returned at the end of Chapter 5 to a few situations from everyday life in order to apply some of the things you had learned from your study of reasoning in geometry. On pages 161-163 you examined ten statements concerning non-mathematical situations in an effort to discover instances of faulty reasoning. In some cases you discovered that the conclusion as stated did not follow from what was given. In other cases the statement contained an assumption that you were unwilling to accept. In still
other cases the argument depended upon the meaning of a term that was not defined.

You returned to the study of geometry for four more chapters and now are asked to consider some more situations from everyday life. From your study of geometry you have acquired an idea of the nature of an abstract logical system. You have been reasoning, and at the same time have been considering what reasoning is. The abstract, orderly, and impersonal propositions of geometry have afforded you an ideal of logical argumentation which ought to help you in your reasoning about the concrete, involved, and personal situations of everyday life. Often in situations of this latter sort there is difference of opinion with respect to the underlying assumptions. Furthermore, from many of the assumptions only a probable consequence can be inferred, not an inevitable consequence. In a problem from real life there is much to be said on each side. We cannot demonstrate that our reasoning is surely right, or surely wrong, as in geometry. We have seen that the truth of geometric theorems is relative to the assumptions on which they rest. The truth of non-mathematical propositions in real life is much less certain. This will be evident from consideration of the following exercises.

**EXERCISES**

The following questions ask you to decide upon a course of action or to form an opinion. In most of these situations you will find it difficult to come to a decision. You are not expected to discover "the correct answer," because the correctness of the answer must be judged in terms of the assumptions that you make. Instead, you are asked to come to a decision and then to record briefly the assumptions you made and the course of reasoning
you followed in arriving at your decision. Sometimes the
decision will turn upon the meaning of a word or phrase.
If so, mention the word and state how you defined it.

Then, in class, compare your answer to each question
with the answers of your classmates and try to discover
the origin of the differences between your answer and
each of the answers that differ from it. Note whether
the disagreement involves an assumption, the definition
of a term, or the method of reasoning.

Even though it is impossible to make your definitions
and assumptions in these non-mathematical situations as
precise as they can be made in geometry, it is hoped that
your study of geometry has given you an ideal of a logical
system that will make you more sensitive to the important
part played by definitions and assumptions in arguments
on non-mathematical questions. If you observe that
people argue questions of everyday life from different as­
sumptions-usually not explicitly stated-and attach dif­
derent meanings to words, you will be ready to turn their
attention first to the need of agreeing on a common start­
ing-point of the argument and then to the need of reason­
ing correctly from point to point.

1. If the boy who sits behind you persists in kicking
you and poking you slyly when the teacher is not looking,
ought you to turn around and pound him then and there,
or wait until school is over? If you wait until school is
over, he will probably have his gang with him; if you hit
him now in class, the teacher will see you and will prob­
ably punish you. What should you do?

2. Late one afternoon after basketball practice three
of the best players on the team broke into the school
lunch room and took some chocolate bars which they
passed around among their cronies. In the investigation
which followed, the principal quickly discovered that the
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break occurred at a time when only the members of the basketball team and a few of their friends were in the building. It was just before the most important games of the season, and it seemed likely that no more games would be played until the lunch-room episode had been cleared up. A friend of one of the players, therefore, was induced by his mates to stand forth as the sole culprit and take all the blame upon himself.

Should he have done so?

What ought the real culprits to do in a case of this sort?

What ought the captain of the team to do?

What should the teacher who was coaching the team do?

What should the principal do?

If the principal cancels a game for such a reason, will the school be publicly disgraced?

3. Shall I leave school to take a good job, or shall I stay and try to get a diploma? If I start working now after but one year in high school, I shall have a head start of three years of experience in the business world over my classmates who stay in high school and finish the course. But on the other hand they may learn things in high school which will help them in business so that after several years they will have much better jobs than I.

4. I wish I could make up my mind whether to go to Arcadia or to some much larger college.

If I go to Arcadia College, I shall have lots of friends to begin with and can get more out of the social side of college life—at least during the earlier years. But, after all, I know these friends pretty well, and while I do not wish to lose them, I shall wish also to make other friends from other parts of the country. Arcadia is not very big and has all the advantages of a small college where classes are small and teachers and pupils get to know each other intimately. Its departments of botany and
mining, moreover, are second to none in the country. I doubt, however, that I wish to specialize in either of these departments, though to be sure I have never studied either botany or mining and have no way of knowing whether I should like them or not. If I go to a big university, I shall be lost in the crowd; but, on the other hand, I can have as teachers some of the leading professors in the country, and I can have access to libraries, laboratories, museums, good music, and theaters such as Arcadia can never hope to have. I wish I knew which to choose.

5. Should those who wish to begin work an hour earlier in summer than in winter simply agree among themselves to do so, or should we set all the clocks ahead one hour in order to encourage everybody to get up earlier in the morning?

6. Is it ever justifiable to lie to a person who is critically ill?

7. "If a thing is worth doing at all, it is worth doing well."

8. Helen's parents rented their house for the summer to friends from Meriden and took Helen to Europe. Phyllis, a friend of Helen, agreed to keep Helen's bicycle for her while she was away. Phyllis rode the bicycle to the circus grounds, and it was stolen. Is she responsible for the loss of the bicycle?

9. The Greens lost their dog, valued at $150, and advertised their loss in a newspaper, offering a reward of $25. Fred found the dog and returned it to the Greens. He learned later that a reward had been offered. Can he collect it?

10. Mr. Oakley, a dealer in coal, firewood, and fuel oil, delivered a cord of firewood by mistake at the home of 272
Mr. Burnham. Mr. Burnham knew that he had ordered no firewood but burned it nevertheless. Must he pay for it?

11. Should the manager of a corporation “tip off” some of his associates in the business that the next dividend to be announced will be greatly in excess of earlier dividends, so that they can buy up stock in anticipation of the rise in price?

12. Should murderers be sentenced to death or to imprisonment for life?

DEDUCTION AND INDUCTION

In this book all our reasoning, whether in mathematical or in everyday situations, has proceeded from undefined terms, definitions, and assumptions. Starting from these we have proved a chain of theorems and have come to recognize the entire structure of undefined terms, definitions, assumptions, and theorems as constituting an abstract logical system. In such a system we say that each proposition is derived from its predecessor by the process of logical deduction. This process of logical deduction is scientific reasoning.

This scientific reasoning must not be confused with the mode of thinking employed by the scientist when he is feeling his way toward a new discovery. At such times the scientist, curious—let us say—about the sum of the angles of a triangle, proceeds to measure the angles of a great many triangles very carefully. In every instance he notices that the sum of the three angles is very close to 180°; so be hazards a shrewd guess that this will be true of every triangle he might draw. This method of deriving a general principle from a limited number of special instances is called induction.
The method of induction always leaves the possibility that further measurement and experimentation may necessitate some modification of the general principle, or perhaps even its complete abandonment. The method of deduction is not subject to upsets of this sort. When in this geometry we deduced from the five basic principles the proposition concerning the sum of the angles of a triangle, our argument convinced us that this proposition must be true for every triangle. This is because Case 1 of Similarity was assumed to be true for every triangle.

When the mathematician is groping for new mathematical ideas, he uses induction. On the other hand, when he wishes to link his ideas together into a logical system, he uses deduction. The laboratory scientist also uses deduction when he wishes to order and classify the results of his observations and his inspired guesses and to arrange them all in a logical system. While building this logical system he must have a pattern to guide him, an ideal of what a logical system ought to be. The simplest exposition of this ideal is to be found in the abstract logical system of demonstrative geometry.

From the preceding discussion it is clear that both deductive and inductive thinking are very useful to the scientist. Our geometry has called almost entirely for deductive thinking. By way of contrast let us consider a few exercises that concern inductive thinking.

**EXERCISES**

1. Can you be absolutely sure from your previous experience that every time you drop a stone from a height it will invariably fall to the ground?

2. Does the gradual disappearance of ships at sea prove beyond question that the earth is a sphere?
3. Actual experiment shows that milk sours more rapidly when it is left uncovered than when it is covered, and that it sours more rapidly when it is left in a warm place than when it is chilled. From these facts, what do you infer is the best way to care for milk?

4. Devise an experiment to show whether bread becomes moldy sooner when kept warm or cold, moist or dry, covered or uncovered.

5. A French scientist, Louis Pasteur, discovered that if a sugar solution were kept in a sterile flask away from all contact with air, the solution would not ferment. He discovered further that if the liquid were later exposed to air in thickly settled regions it almost always fermented, whereas if it were exposed to air on a mountain top it was less likely to ferment. Would you argue from this that the air alone caused the fermentation? If so, how do you account for the fact that the air does not invariably cause fermentation?

6. If the ferment which causes milk to sour cannot thrive at low temperatures and is killed by high temperatures, what would you do to prevent milk from souring? Is it sufficient to kill the bacteria in milk, or must you also take pains to prevent fresh bacterial growth resulting from contamination by the air? How would you do this? Look in the dictionary for the meaning of the word "pasteurize."

7. Why are preserve jars filled to overflowing before being capped? Why are they then heated again?

8. Lister, a British surgeon, was the first to cover his patients' wounds with gauze soaked in carbolic acid. This admitted air to the wounds but prevented the formation of pus. Why?
9. If two groups of people eat exactly the same amounts of the same foods on a certain day, except that every member of one group eats ice-cream while every member of the other group declines it, and if subsequently every member of the first group becomes very ill, though no member of the second group does, where should you look first for the cause of the illness?

10. If two other groups of people eat wholly different foods for a day except for some canned lobster, which they share, and if subsequently both groups become very sick with identical symptoms, where should you look first for the cause of their sickness?

Inquiries like these in Exercises 9 and 10 above guide the doctor in his diagnosis, lead the detective to discover the perpetrator of the crime, and reveal to the scientist the general law governing the phenomena he has been observing. The doctor, the detective, and the scientist tend to reason as follows:

If the circumstances surrounding two dissimilar events have all but one item in common, then the item with respect to which they differ probably accounts for the dissimilarity between the two events.

If the circumstances surrounding two similar events differ in every respect but one, then that single item which they have in common is probably the cause of the two events, or is intimately connected with it.

When a scientist is searching for a theory that will explain the results of his experiments, he makes assumptions and from them tries to derive propositions that are in accord with the facts as he has found them. He promptly rejects assumptions leading to propositions which disagree with his findings. If he assumes that a moth will eat only woolen goods that are dirty or greasy, then it follows that a clean woolen bathing suit should
never suffer from moths. But if the scientist discovers a clean woolen bathing suit that has been eaten by moths, he immediately rejects his first assumption and adopts a new one to the effect that a moth will eat anything woolen.

The scientist has two problems—one, to discover new scientific propositions; the other, to devise a set of assumptions, or general principles, from which all his propositions can be logically deduced. He recognizes, of course, that propositions are true only in the sense that the assumptions underlying them are assumed to be true. But he judges the usefulness of a set of assumptions by the propositions that can be deduced from them. The best set of assumptions from his point of view is that set which is simplest and leads most directly to the propositions he is interested in. Similarly, in geometry, that set of assumptions is best that is simplest and leads most directly to the most important theorems. Let us look again at the assumptions underlying our geometry.

Whenever we have proved a theorem in this geometry, we have shown in effect that it is a logical consequence of the propositions that were assumed at the outset. In this sense only can we assert that a theorem is "true." As for the truth of the propositions that were taken as fundamental assumptions, we can say only that they are assumed to be true. Some of them may seem obvious to us; others may seem very queer. But so long as they do not contradict one another they can serve as basis for the logical development of theorems.

There are strange geometries, called non-Euclidean geometries, that deny that through a given point not on a given line there is one and only one parallel to the given line. In one of these non-Euclidean geometries every straight line intersects every other straight line—there are no parallel lines. A second non-Euclidean geometry admits many parallels through a given point to
a given line. These geometries contain theorems that seem very queer to us—for example, that the sum of the angles of a triangle is always greater, or less, than 180 degrees. But these theorems are true in terms of the assumptions from which they were derived, and there is no contradiction within the assumptions themselves. The queerness of these theorems springs from the fact that at least one of the assumptions of each of these geometries contradicts our everyday observation and experience.

In the geometry that we have been studying, however, not only are all the assumptions in accord with our experience, and therefore "obvious," but some of the theorems appear to be obvious also. This apparent obviousness is no guarantee of the truth of these theorems. In a few instances, however, we have ignored this fact and have allowed ourselves the privilege of skipping the proofs of certain theorems that seemed so obvious that we did not want to bother to prove them. This amounted, in effect, to adding to our list of assumptions. Now that we see the logical pattern of this geometry more clearly, let us consider again the assumptions underlying it and try to make the list of our assumptions as short as possible.

**EXERCISES**

4. We have stated three Cases of Similarity, taking Case 1 for granted and deriving Cases 2 and 3 from it. Show that we could have assumed Case 2 instead, deriving Cases 1 and 3 therefrom.

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6. Prove Theorem 13, page 107, if you omitted the proof earlier in the course.

7. Consider the propositions that were necessary to help us prove Theorem 13, and record the propositions from which each of these was derived. Continue this, tracing back the links of this logical chain until you have established the dependence of Theorem 13 upon Principle 5.

8. In similar manner trace back the links of the logical chain that binds Theorem 16, page 111, to Principle 5.

9. Trace the logical chain that connects Theorem 20, page 139, with the fundamental principles of this geometry. At least one link of this chain is a definition.

10. Theorem 21, page 139, says in effect that if a straight line has but one point in common with a circle it is perpendicular to the radius drawn to the point of contact. This theorem was proved by showing that if the given line were not perpendicular to the radius drawn to the point of contact it would have more than one point in common with the circle. Justify this procedure in terms of the discussion on page 247.

11. As in Exercises 7 to 9 above, show the logical dependence of Theorem 22, page 145, upon the fundamental principles of this geometry.

12. Show similarly the logical dependence of Theorem 23, page 152, upon our fundamental principles.

13. Study the note on the area of polygons at the end of Chapter 7, page 222. Show how, by starting with the formula $A = \frac{1}{2}bh$ for the area of a triangle, the areas of parallelograms, rectangles, and other polygons can be derived. Notice how easy this is. Notice also that the area of a triangle, $\frac{1}{2}bh$, as developed on page 222, is merely a number. This number is the product of three other numbers, two of which measure distances.
14. Using the improper diagram in Fig. 9, page 32, "prove" that every triangle is isosceles.

15. Using a correct diagram instead of the improper one on page 32, show where the fallacious proof in Ex. 14 breaks down.

**LOGIC AND LANGUAGE**

To avoid pitfalls in logic, we must be very careful of the language we employ, for there are many terms in everyday use that have a technical meaning quite different from their ordinary meaning. One example of this is the word "ruler." To the mathematician a ruler is an unmarked straightedge designed only for ruling. In ordinary usage and for convenience, however, this ruler has a scale marked on it so that it can be used also for measuring distances. In order to avoid every confusion of this sort between ordinary meanings and technical meanings, mathematicians and logicians have deliberately invented a very artificial vocabulary and language for the communication of their ideas.

Mathematicians and logicians do this not only to avoid misinterpretation. They have another object in view as well, which is to keep a body of propositions open as long as possible to several different interpretations. Thus the wording is not prejudiced prematurely in favor of a particular interpretation. A single logical system that is expressed in general logical language in this manner can have several different applications according to the way in which the logical language is interpreted. We shall understand this idea better if we invent a bit of artificial language of our own, as explained on the next page, and then discover some of the ways in which this language can be interpreted.
The meanings of noun, singular, plural, adverb, verb, present, past, future will be assumed to be understood without definition.

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<tr>
<td>blam</td>
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Ideas:
(a) Swog ogsput blam.
(b) Swog lep ogviss blam.
(c) Swoga sput blama,
(d) Swoga ikviss blama.

**EXERCISES**

*1. Convey the above ideas in your own language in which swog means child, blam means cake, lep means not, viss means eat, sput means see.

*2. Convey the above ideas in Nellie's language in which swog means man, blam means boy, lep means not, viss means spank, sput means catch.

*3. Convey the above ideas in another language in which swog means circle, blam means triangle, lep means not, viss means is inside of, sput means is outside of.

*4. Convey the above ideas in a fourth language in which swog means quotient, blam means divisor, lep means not, viss means is less than, sput means is greater than.

**AN ABSTRACT LOGICAL SYSTEM**

This artificial language seems inadequate when we come to express mathematical ideas. Let us try again. We shall take the following as undefined terms: A, B, C, D, ..., K, L, M, ..., X, Y, Z, is, are, not, and, or, but, if, then, all, every, =, -c.
DEFINITIONS: 1. If \( A < B \), then \( B > A \). This defines \( > \).
2. If \( A < X \) and \( X < B \), then \( X \) is between \( A \) and \( B \).

ASSUMPTIONS: 1. \( A < B \), or \( =B \), or \( >B \).
2. If \( A < B \) and \( B < C \), then \( A < C \).
3. If \( A = B \) and \( B = C \), then \( A = C \).

Theorem 1. If \( A < B \) and \( B < C \) and \( C < D \), then \( A < D \).

PROOF: \( A < C \) (Assumption 2); and if \( A < C \) and \( C < D \), then \( A < D \) (Assumption 2).

Theorem 2. If \( A = B \) and \( B < C \), then \( A < C \).

PROOF: If \( A \) is not \( < C \), then \( A = C \) or \( A > C \) (Assumption 1).

If \( A = C \), then \( B = C \) (Assumption 3). But \( B < C \).

If \( A > C \), then \( C < A \) (Definition 1). If \( B < C \) and \( C < A \), then \( B < A \) (Assumption 2).
But \( B = A \).

Therefore \( A \) is not \( = C \); and \( A \) is not \( > C \).
Therefore \( A < C \) (Assumption 1).

EXERCISES

*1. Prove that if \( K < L \), and \( K \) and \( L \) are between \( A \) and \( B \), and \( X \) is between \( K \) and \( L \), then \( X \) is between \( A \) and \( B \).

*2. Interpret the foregoing assumptions and theorems if the elements \( A, B, C, \ldots \) refer to people, if the symbol \( "=\)" means \( "\text{is of the same age as}" \) and if the symbol \( "<\)" means \( "\text{is older than}" \).

*3. Interpret the foregoing assumptions and theorems if the elements \( A, B, C, \ldots \) refer to numbers, if the
symbol "=" means "is equal to," and if the symbol "<" means "is less than."

*4. Interpret the foregoing assumptions and theorems if the elements $A, B, C, \ldots$ refer to points on a line, if the symbol "=" means "coincides with," and if the symbol "<" means "precedes."

At the outset of this chapter your attention was diverted from geometry while you were called upon to reason deductively in a number of non-mathematical situations. You were then shown the contrast between deductive and inductive thinking. Following this, you reconsidered the general logical structure of our geometry. Finally you met an abbreviated abstract logical system and observed how a suitable interpretation of its artificial terminology—in Ex. 4 above—turns this abbreviated abstract logical system into an abbreviated logical system of geometry. Thus you can understand that every logical system of geometry is but an interpretation or concrete illustration of a general system of logic.

The basic purpose of this entire course in demonstrative geometry has been to improve your skill in reasoning and to give you an appreciation of the science of reasoning. Now that you have become acquainted with a method and pattern of thinking that can be applied in the many critical situations of everyday life, you ought—in the years that lie ahead of you—to apply what you have learned to a careful study of some of the serious problems that confront society, and ought then to try to solve these problems by a course of action based on straight thinking.
Laws of Number

This geometry considers the points on a line to be paired with the real numbers. This requires of the pupils only that they take for granted their naive intuitions concerning the real number system; that they take for granted, for example, that if \( a = b \), then \( \frac{a}{2} = \frac{b}{2} \); and, also, that one can approximate to any irrational number in terms of rational numbers as closely as one pleases.

The authors do not consider it necessary or desirable that pupils, when proving theorems, should supply reasons for such statements as "If \( a = b \), then \( \frac{a}{3} = \frac{b}{3} \);" they prefer to trust to the pupils' intuition in cases of this sort. But many teachers have been in the habit of asking pupils to support statements of this sort with the reason "If equals are divided by equals, the quotients are equal," and they wish to continue to do so. They call this an "axiom," and this may well be a good name for it as far as the pupils are concerned. Actually, some of these "axioms" are postulates, others are definitions, and still others are theorems in the development of the real number system; for algebra as well as geometry has a logically ordered structure.

It is very much simpler to introduce a pupil to an abstract logical system through the medium of geometry than through algebra. But the teacher should know that every reference to number in elementary demonstrative geometry requires strictly a justification drawn from the logical development of the algebra of real numbers. Even though he does not mention it to the pupil, he should know by what right he says, "If \( a = b \), then \( \sqrt[3]{a} = \sqrt[3]{b} \)" and "7 is greater than 5." The justification for this latter is to be found in the very definition of the number 7 and not in the oft-quoted but erroneous declaration that "The whole is greater than any of its parts."
In order to show the relation of some of the familiar “axioms” to a logical development of the real number system, the first steps of such a development are given below, paraphrased in some instances for ready reference. Certain statements in this development are admittedly somewhat forbidding even to teachers. It is hoped, however, that there will be many who will be glad to have the horrid details gathered together in one place.

We take the ideas of number, order, and equal as undefined and assume the postulates necessary to establish the real number system, as follows. A few of these postulates are stated in two forms; the second, in words instead of symbols, is a familiar phrasing which some teachers find convenient.

The symbols $=, <, +, \times$ are undefined. They acquire meaning, however, from the following postulates governing their use. We read them “equals,” “precedes,” “plus,” “times,” respectively. We shall need also the undefined terms commonly employed in every sort of logical reasoning, as, for example, is, are, not, and, or, but, if, then, all, every.

The letters represent real numbers, like 5, -2.1, 1 37, $\sqrt{2}$, $-\sqrt{4}$, $\pi$.

1. $a = a$.

2. If $a = b$, then $b = a$.

3. If $a = b$ and $b = c$, then $a = c$. (Numbers that are equal to the same number or to equal numbers are equal to each other.)

4. If $x = y$, then $f(x, a, b, c, \ldots) = I(y, a, b, c, \ldots)$, where the expression $f(x, a, b, c, \ldots)$ denotes a real number built up from successive combinations of the numbers $x, a, b, c, \ldots$ and the operations $+$ and $\times$, and $I(y, a, b, c, \ldots)$ denotes the number obtained from $I(x, a, b, c, \ldots)$ by writing $y$ in place of $x$ throughout. (A number may be substituted for its equal in an equation or in an inequality.)

5. If $a$ and $b$ are real numbers, then $a + b$ is a real number uniquely determined by $a$ and $b$. Sometimes, for convenience, we write it in parentheses, as $(a + b)$. 

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6. \(a + b = b + a\).

7. \((a + b) + c = a + (b + c)\).

8. If \(a + b = a + c\), then \(b = c\).

9. There is a real number \(z\) such that \(z + z = z\). It can be proved that there is only one such number \(z\). We shall call it "zero," by definition, and shall denote it by the symbol \(0\).

10. Corresponding to every real number \(a\) there is a real number \(a'\) such that \(a + a' = 0\). It can be proved that there is only one such number \(a'\). We shall call it the "negative" of \(a\) and shall denote it by the symbol \(-a\).

As a new definition we shall write \(b + (-a) = b - a\), where the symbol \(-\) read "minus," on the right of the equation, denotes a new operation which is defined by this equation.

11. If \(a\) and \(b\) are real numbers, then \(a \times b\) is a real number uniquely determined by \(a\) and \(b\). Sometimes, for convenience, we write it in parentheses, as \((a \times b)\).

12. \(a \times b = b \times a\).

13. \((a \times b) \times c = a \times (b \times c)\).

14. If \(a \times b = a \times c\), then \(b = c\), provided \(a\) is not 0.

15. There is a real number \(w\) different from 0 such that \(w \times w = w\). It can be proved that there is only one such number \(w\). We shall call it "one," by definition, and shall denote it by the symbol \(1\).

The numbers 0, 1, 1+1 = 2, 2+1 = 3, ... and -1, \((-1) + (-1) = -2\), ... are called "integers."

16. Corresponding to every real number \(a\) except 0 there is a real number \(a'\) such that \(a \times a' = 1\). It can be proved that there is only one such number \(a'\). We shall call it the "reciprocal" of \(a\) and shall denote it by the symbol \(\frac{1}{a}\).

As a new definition we shall write \(b \times \frac{1}{a} = b \div a\), where the symbol \(\div\), read "divided by," on the right of the equation, denotes a new operation which is defined by this equation.
The quotient of any two integers, \( m \) and \( n \), provided \( n \) does not equal 0, is called a "rational number." It is usually written \( \frac{m}{n} \).

17. If \( a, b, c \) are real numbers, then \( aX(b+c) = (aXb) + (axc) \).

18. If \( a \) is not equal to \( b \), then either \( a-cb \) or \( b-ae \).

19. If \( a<\)\( b \), then \( a \) is not equal to \( b \).

20. If \( a-\)\( cb \) and \( b-\)\( ee \), then e-\( cc \). (If one real number precedes a second, and if this second number precedes a third, then the first precedes the third.)
   If \( a<x \) and \( x-\)\( cb \), then \( x \) is said to be "between" \( a \) and \( b \), and also "between" \( b \) and \( a \).

21. If all the real numbers are arranged in order according to the relation \( < \), then every separation of this totality of real numbers into two classes \( C_1 \) and \( C_2 \) such that every real number belongs either to \( C_1 \) or to \( C_2 \) and such that every real number in \( C_1 \) precedes every real number in \( C_2 \) determines a real number \( s \) such that every real number which precedes \( s \) belongs to \( C_1 \) and every real number which \( s \) precedes belongs to \( C_2 \); the real number \( s \) is either the last number in \( C_1 \) or the first number in \( C_2 \).

22. The totality of real numbers is such that between any two real numbers there is at least one rational number.

23. If \( a, x, \) and \( y \) are any real numbers, and if \( x<y \), then \( a+x<y \).

24. If \( a \) and \( b \) are any two real numbers such that \( O<\)\( a \) and \( O<\)\( b \), then \( O<\)\( axb \).

From the foregoing postulates the rest of the real number system can be built up. In particular, we can infer the following statements.

If equal numbers are added to equal numbers, the sums are equal.

The negatives of equal numbers are equal.

If equal numbers are subtracted from equal numbers, the remainders are equal.
If equal numbers are multiplied by equal numbers, the products are equal.

The reciprocals of equal numbers are equal. (But zero has no reciprocal.)

If equal numbers are divided by equal numbers, the quotients are equal. We cannot divide by zero, however.

Equal (positive integral) powers of equal numbers are equal.

Equal (positive integral) roots of equal numbers are equal.

If equal numbers are added to unequal numbers, the sums are unequal in the same order.

If equal numbers are subtracted from unequal numbers, the remainders are unequal in the same order.

If unequal numbers are subtracted from equal numbers, the remainders are unequal in the opposite order.

If unequal positive numbers are multiplied by equal positive numbers, the products are unequal in the same order.

If unequal positive numbers are divided by equal positive numbers, the quotients are unequal in the same order.

If equal positive numbers are divided by unequal positive numbers, the quotients are unequal in the opposite order.

If unequal numbers are added to unequal numbers in the same order, the sums are unequal in the same order.

If unequal positive numbers are multiplied by unequal positive numbers in the same order, the products are unequal in the same order.
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